MOTIVIC OBSTRUCTION TO RATIONALITY OF A VERY GENERAL CUBIC HYPERSURFACE IN $\mathbb{P}^5$

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Abstract. Let S be a smooth projective surface over a field. We introduce the notion of integral decomposability and, respectively, the opposite notion of integral indecomposability, of the transcendental motive $M^2_{\mathrm{tr}}(S)$. If the transcendental motive is indecomposable rationally, then it is indecomposable integrally. For example, $M^2_{\mathrm{tr}}(S)$ is rationally, and hence integrally indecomposable if S is an algebraic $K3$-surface whose motive is known to be finite-dimensional. In the paper we prove that $M^2_{\mathrm{tr}}(S)$ is integrally indecomposable when S is the self-product of a smooth projective curve having enough morphisms onto an elliptic curve with complex multiplication. This applies, for example, when S is the self-product of the Fermat sextic in $\mathbb{P}^2$. Some refinement of the same technique yields that $M^2_{\mathrm{tr}}(S_6)$ is integrally indecomposable, where S_6 is the Fermat sextic in $\mathbb{P}^3$. This suggests a conjecture saying that the transcendental motive of any smooth projective surface is integrally indecomposable. We prove in the paper that if this motivic integral indecomposability conjecture is true, and if the motive of any smooth projective surface is finite-dimensional, then a very general cubic hypersurface in $\mathbb{P}^5$ is not rational.

Contents

1. Introduction

A well-known conjecture in algebraic geometry says that a very general cubic hypersurface in $\mathbb{P}^5$ is not rational. Since such fourfolds are unirational, the conjecture is a particular case of the Lüroth problem. Whereas the Lüroth problem for cubic threefolds was solved by means of abelian invariants, [14], the numerous attempts to develop an analog of the Clemens-Griffiths theory, which would be

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appropriate in dimension 4, have not achieved the desired result yet. The reason for that is possibly rooted in the existence of phantom subcategories discovered in [9], [10] and [18].

A well-known birational invariant of cycle-theoretic nature is the Chow group of 0-cycles modulo rational equivalence on a variety over a non-algebraically closed field. The recent developments along this line include the notion of CH_0-triviality introduced in [3]. In [36] Voisin proved that CH_0-nontriviality is a deformable property in families, and used this to prove the stable non-rationality for the desingularization of a very general quartic double solid with at most seven nodes. In [13] Colliot-Thélène and Pirutka used similar method to prove the existence of not stably rational smooth quartic hypersurfaces in $\mathbb{P}^4$.

However, as we do not know a single example of a nonrational cubic fourfold in $\mathbb{P}^5$, it is not clear how to use the deformation of CH_0-nontriviality in the striking dimension 4 case. Our aim in this paper is to develop a motivic obstruction to rationality of a very general cubic fourfold in $\mathbb{P}^5$, which would avoid the difficulties above. There are two advantages of the motivic approach presented in this paper. The first one is that there is no phantom submotives in a motive, provided it is finite-dimensional, see Proposition 7.5 in [22]. The second advantage is that the obstruction to rationality of a fourfold is given in terms of rational equivalence of 0-cycles on surfaces, rather than on the fourfold itself.

To explain the idea, let X be a smooth projective connected variety of dimension n over a field, and let $CH^n(X \times X)$ be the Chow group of codimension n algebraic cycles modulo rational equivalence on $X \times X$, with coefficients in $\mathbb{Z}$. Recall that an algebraic cycle class $\Xi \in CH^n(X \times X)$ is said to be balanced, if Ξ is a sum of classes represented by algebraic cycles supported on $Y \times X$ or $X \times Z$, where Y and Z are closed subschemes of positive codimension in X. We will say that Ξ is *essential*, if it is not torsion, not numerically trivial and not balanced in $CH^n(X \times X)$. The motive $M(X)$ is said to be *integrally essentially decomposable*, if the diagonal class Δ of the variety X can be presented as a sum of two orthogonal essential idempotents in $CH^2(X \times X)$. Otherwise, $M(X)$ is *integrally essentially indecomposable*. For example, the motive of a smooth projective curve is essentially indecomposable.

If S is a smooth projective surface over a field, its Albanese kernel is controlled by the transcendental motive $M^2_{\mathrm{tr}}(S)$ introduced in [21]. Although $M^2_{\mathrm{tr}}(S)$ lives in the category of Chow motives with coefficients in $\mathbb{Q}$, integral essential (in)decomposability of the entire motive $M(S)$ can be viewed as integral (in)decomposability of the transcendental motive $M^2_{\mathrm{tr}}(S)$.

Clearly, if $M^2_{\mathrm{tr}}(S)$ is indecomposable rationally, then it is indecomposable integrally. For example, if S is an abelian surface isogenous to the self-product of an elliptic curve with complex multiplication over a field of characteristic 0, then $M^2_{\mathrm{tr}}(S)$ is rationally, and hence integrally indecomposable. The same is true if S is an algebraic $K3$-surface over $\mathbb{C}$ whose motive is finite-dimensional, as the transcendental Hodge structure is indecomposable by [38] and finite-dimensional motives have no phantom submotives by [22]. In particular, the transcendental motive of the resolution of the Kummer quartic, the Fermat quartic or any quartic of Weil type in $\mathbb{P}^3$ is integrally indecomposable.

The following theorem gives more examples of surfaces whose transcendental motives decompose rationally but they are indecomposable integrally.

Theorem A. *Let C be a smooth projective curve over a field k of characteristic 0. Assume that there is a finite group G of automorphisms of the curve C, and nonconstant regular morphisms,*

$$\phi_i : C \to E \ , \qquad i = 1, \dots, r \ ,$$

where E is an elliptic curve with complex multiplication over k, one for each irreducible representation V_i of the action of G on $H^0(\Omega_C)$, such that the image of the pullback homomorphism

$$\phi_i^* : H^0(\Omega_E) \to H^0(\Omega_C)$$

is in V_i. Then the motive $M^2_{\mathrm{tr}}(C \times C)$ is integrally indecomposable, if $\deg(\phi_i) \geq 4$ for all i.

An explicit example of a curve satisfying the assumptions of Theorem A is the Fermat sextic C_6 in $\mathbb{P}^2$, see the proof of Proposition 7 in [7]. A refinement of the technique used in the proof of Theorem A gives us the following result, which should be compared with the main result in [2].

Theorem B. *Let S_6 be the Fermat sextic in $\mathbb{P}^3$. Then the motive $M^2_{\mathrm{tr}}(S_6)$ is integrally indecomposable.*

The meaning of Theorem B is that, even if the transcendental Hodge structure of a smooth projective surface is integrally decomposable, yet its transcendental motive can be integrally indecomposable. This suggests that the following motivic analog of Kulikov's Hodge-theoretic indecomposability conjecture, [24], may be true.

Motivic indecomposability conjecture. *The transcendental motive of a smooth projective surface over a field of characteristic 0 is integrally indecomposable.*

Now recall that the well-know conjecture due to Kimura and O'Sullivan asserts that all Chow motives are finite-dimensional. Our third theorem is conditional.

Theorem C. *If the motivic indecomposability conjecture is true, and if the motive of any smooth projective surface is finite-dimensional, then a very general cubic fourfold hypersurface in $\mathbb{P}^5$ is not rational.*

It should be pointed out here that, as it was recently announced, Ayoub's conservativity conjecture for the de Rham, and hence Betti realization of Voevodsky's geometric motives is now proven, see [5] and [6]. If this is indeed the case, then the motives of smooth projective surfaces are finite-dimensional, see Corollary 2.14 in [4], and therefore the cubic fourfold non-rationality conjecture follows solely from the motivic indecomposability conjecture above.

The paper is organized as follows. The next Section 2 is written merely for those readers who feel uncomfortable with Chow motives and the Chow-Künneth decompositions. Section 3 is devoted to the notion of integral essential (in)decomposability, and we briefly discuss the integral essential indecomposability of the motives of products of elliptic curves with complex multiplication and $K3$-surfaces. In Section 4 we prove Theorem A. In Section 5 we study in detail the Fermat sextics and prove Theorem B. Finally, in Section 6, we state the motivic indecomposability conjecture and prove the conditional Theorem C.

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2. Preliminaries and notation

For an algebraic scheme X over a field, let $CH_r(X)$ be the Chow group of dimension r algebraic cycles modulo rational equivalence on X. Let also $A_r(X)$ be the subgroup generated by algebraically trivial cycle classes in $CH_r(X)$. If X is equidimensional of dimension n, then we write $CH^{n-r}(X)$ and $A^{n-r}(X)$ instead of $CH_r(X)$ and $A_r(X)$ respectively. One may also speak about R-modules $CH^j(X)_R$ and $A^j(X)_R$, where R is a commutative ring of characteristic 0 and, for an abelian group A, A_R is the tensor product of A and R over $\mathbb{Z}$.

Let k be a field. The category of Chow motives $\mathsf{C}(k)$ over k will be contravariant, i.e. if X and Y are two smooth projective varieties over k, and $X = \cup_j X_j$ is the decomposition of X into connected components, then the group $CH^m(X,Y)$ of correspondences of degree m from X to Y is the direct sum of the groups $CH^{n_j+m}(X_j \times Y)$, where n_j is the dimension of the component X_j. For any two correspondences $\alpha \in CH^m(X,Y)$ and $\beta \in CH^n(Y,Z)$ their composition $\beta \circ \alpha$ is the correspondence $p_{13*}(p_{12}^*(\alpha) \cdot p_{23}^*(\beta))$, where the central dot stays for the intersection of cycle classes and the projections are obvious. The correspondence $\beta \circ \alpha$ is an element of the group $CH^{m+n}(X,Z)$.

The objects of $\mathsf{C}(k)$ may be conceived as triples (X, Σ, m), where Σ is an idempotent[1] in the algebra $CH^0(X,X)$, and m is an integer. For two motives $M = (X, \Sigma, m)$ and $N = (Y, \Xi, n)$, the group $\mathrm{Hom}_{\mathsf{C}(k)}(M,N)$ consists of all triple compositions $\Xi \circ \Phi \circ \Sigma$, where $\Phi \in CH^{n-m}(X,Y)$. The transposed graphs Γ_f^{t} of regular morphisms $f : X \to Y$ are in $CH^0(Y,X)$ and give the standard functor from smooth projective varieties over k to $\mathsf{C}(k)$. The graph of the identity map for X is the diagonal class $\Delta \in CH^0(X,X)$. The motive $M(X)$ is the triple

[1]throughout the paper the words "idempotent" and "projector" are synonyms

$(X, \Delta, 0)$. If Σ is an idempotent in $CH^0(X, X)$, it is convenient to write M_Σ instead of the triple $(X, \Sigma, 0)$.

The category $\mathsf{C}(k)$ is symmetric monoidal, where the monoidal product of two motives (X, Σ, m) and (Y, Ξ, n) is the motive $(X \times Y, \Sigma \otimes \Xi, m + n)$. The triple $\mathbb{1} = (\mathrm{Spec}(k), \Delta, 0)$ is the monoidal unit. The triple $\mathbb{L} = (\mathrm{Spec}(k), \Delta, -1)$ is called the Lefschetz motive over k. Clearly, the motive $M(\mathbb{P}^1)$ is a direct sum of the unit $\mathbb{1}$ and the Lefschetz motive $\mathbb{L}$. We will be also using the Tate motive $\mathbb{T} = \mathbb{L}^{-1} = (\mathrm{Spec}(k), \Delta, 1)$, i.e. the monoidal inverse to $\mathbb{L}$ in $\mathsf{C}(k)$.

The category $\mathsf{C}(k)_R$ with coefficients in R is obvious. Apart from the integral category $\mathsf{C}(k)$, within this paper we will need the categories of Chow motives $\mathsf{C}(k)_{\mathbb{Q}}$ and $\mathsf{C}(k)_{\mathbb{Z}[1/n]}$, where n is a positive integer and $\mathbb{Z}[1/n]$ is the ring obtained by inverting the powers of the number n.

In the same vein, one can also define the groups $N_r(X)$ of algebraic r-cycles modulo numerical equivalence on X, and construct the category $\mathsf{N}(k)$ of pure motives modulo numerical equivalence over k. The category $\mathsf{N}(k)_{\mathbb{Q}}$ is rigid tensor and $\mathbb{Q}$-linear. Moreover, it is known to be semisimple abelian by Jannsen's result, see [19]. If Σ is a cycle class modulo rational equivalence on a variety X over k, we will write $\bar{\Sigma}$ for its class modulo numerical equivalence on X. If $M = (X, \Sigma, m)$ is a Chow motive, then $\bar{M} = (X, \bar{\Sigma}, m)$ is the corresponding numerical motive over k. The functor from $\mathsf{C}(k)$ to $\mathsf{N}(k)$ sending M to $\bar{M}$ is tensor, and the same with coefficients in R. The following lemma will be systematically applied in the context of the abelian semisimple category $\mathsf{N}(k)_{\mathbb{Q}}$.

Lemma 1. *Let $\mathscr{A}$ be a semisimple abelian category, let X be an object in $\mathscr{A}$, and let*

$$\mathrm{id}_X = a + b$$

and

$$\mathrm{id}_X = e_1 + \ldots + e_n$$

be two different decomposition of the identity automorphism of X in to two sets of pairwise orthogonal idempotents in the associative ring $\mathrm{End}(X)$*. Assume, moreover, that the images of all the idempotents $e_1, \ldots, e_n$ are simple objects in the category $\mathscr{A}$. Then the set of indices*

$$I = \{1, \ldots, n\}$$

can be represented as a disjoint union of two subsets

$$I = J \sqcup K \ ,$$

such that

$$a = \sum_{i \in J} e_i \qquad \text{and} \qquad b = \sum_{i \in K} e_i \ .$$

Proof. Since

$$a + b = \mathrm{id}_X = e_1 + \ldots + e_n$$

and a and b are orthogonal idempotents, multiplying by a yields

$$a = a^2 = a(a + b) = ae_1 + \ldots + ae_n \ .$$

For each index $i \in I$ let

$$X \xrightarrow{e'_i} M_i \xrightarrow{e''_i} X$$

be the decomposition of the idempotent e_i through its image. Then

$$e'_i e''_i = \mathrm{id}_{M_i} ,$$

whence the morphism e''_i is a monomorphism, and the morphism e'_i is an epimorphism in $\mathscr{A}$.

Similarly, let

$$X \xrightarrow{a'} A \xrightarrow{a''} X$$

and

$$X \xrightarrow{b'} B \xrightarrow{b''} X$$

be the decompositions of a and, respectively, b through their images, so that

$$a'a'' = \mathrm{id}_A ,$$

$$b'b'' = \mathrm{id}_B ,$$

and hence a'' and b'' are monomorphisms and a' and b' are epimorphisms in $\mathscr{A}$. Then, for any index $i \in I$, whether the composition ae_i is 0 or not depends on the same question for the composition $a'e''_i$, and similarly for compositions be_i.

As the category $\mathscr{A}$ is abelian semisimple, the object A decomposes into simple objects,

$$A = A_1 \oplus \dots A_s ,$$

and the object B decomposes into simple objects,

$$B = B_1 \oplus \dots B_t .$$

Let

$$a = a_1 + \dots + a_s \qquad \text{and} \qquad b = b_1 + \dots + b_t$$

be the corresponding decompositions of the idempotents a and b into mutually orthogonal idempotents, and let

$$A \xrightarrow{a'_j} A_j \xrightarrow{a''_j} A$$

and

$$B \xrightarrow{b'_k} B_k \xrightarrow{b''_k} B$$

be the decompositions of the idempotents through their images.

Then, for each index $i \in I$, the composition $a'e''_i$ is 0 if and only if the composition $a'e''_i$ is 0. The latter holds if and only if there exists an index j in $J = \{1, \dots, s\}$, such that the composition $a''_j a' e''_i$ is 0. But as the objects M_i and A_j are simple, the composition $a''_j a' e''_i$ is either 0 or an isomorphism.

Similarly, for each index $i \in I$, the composition $b'e''_i$ is 0 if and only if the composition $b'e''_i$ is 0. The latter holds if and only if there exists an index k in $K = \{1, \dots, t\}$, such that the composition $b''_k b' e''_i$ is 0. But as the objects M_i and B_k are simple, the composition $b''_k b' e''_i$ is either 0 or an isomorphism.

Since X is the direct sum of A and B, the same object M_i cannot be inside A and B at the same time. Therefore, for each index $i \in I$ either there exists an index $j \in J$ such that $a''_j a' e''_i$ is an isomorphism, and then $b''_k b' e''_i$ is 0 for all

$k \in K$, or there exists an index $k \in K$ such that $b''_k b' e''_i$ is an isomorphism, and then $a''_j a' e''_i$ is 0 for all $j \in J$.

This gives the obvious decomposition

$$I = J \sqcup K$$

of the set I into two disjoint subsets, where

$$J = \{i \in I \mid ae_i \neq 0 \text{ but } be_i = 0\}$$

and

$$K = \{i \in I \mid be_i \neq 0 \text{ but } ae_i = 0\} .$$

And since

$$a = ae_1 + \ldots + ae_n ,$$

we obtain that

$$a = \sum_{i \in J} ae_i .$$

Similarly,

$$b = be_1 + \ldots + be_n ,$$

and hence

$$b = \sum_{i \in K} be_i .$$

Moreover, if $i \in J$ then

$$ae_i = ae_i + 0 = ae_i + be_i = (a+b)e_i = \mathrm{id}_X e_i = e_i ,$$

and, similarly, if $i \in K$ then

$$be_i = 0 + be_i = ae_i + be_i = (a+b)e_i = \mathrm{id}_X e_i = e_i .$$

Therefore,

$$a = \sum_{i \in J} ae_i = \sum_{i \in J} e_i$$

and, similarly,

$$b = \sum_{i \in K} be_i = \sum_{i \in K} e_i .$$

□

For any prime l different from the characteristic of k, and any field extension L/k, let $H^j_{ét}(X_L, \mathbb{Q}_l(i))$ be the j-th l-adic étale cohomology group of a variety X_L over L twisted by i. If L is the algebraic closure $\bar{k}$ of the ground field k, such étale cohomology groups provide a Weil cohomology theory over k. In particular, for any smooth projective X over k there is a cycle class homomorphism from $CH^j(X)$ to $H^{2j}_{ét}(X_{\bar{k}}, \mathbb{Q}_l(j))$, whose kernel will be denoted by $CH^j(X)_{\mathrm{hom}}$.

If L is a field extension of k and there exists an embedding $\sigma : L \hookrightarrow \mathbb{C}$ over k, each embedding $\bar{\sigma} : \bar{L} \hookrightarrow \mathbb{C}$ over σ gives the pullback isomorphism between the étale cohomology groups $H^{2p}_{ét}(X_{\bar{L}}, \mathbb{Q}_l(p))$ and $H^{2p}_{ét}(X_{\mathbb{C}}, \mathbb{Q}_l(p))$, commuting with the cycle class maps. The latter group is isomorphic to the Betti cohomology group $H^{2p}(X_{\mathbb{C}}, \mathbb{Q}_l)$ with coefficients in $\mathbb{Q}_l$. Therefore, homological triviality of algebraic cycles is independent on the type of cohomology, and we may write

$H^i(X)$ meaning either l-adic étale cohomology over $\bar{k}$ or Betti cohomology groups over L embeddable into $\mathbb{C}$.

Now, for any smooth projective connected variety X of dimension n over k the class $cl(\Delta)$ in $H^{2n}(X \times X)$ decomposes into the Künneth components $cl(\Delta)_{i,n-i}$, for all $0 \le i \le 2n$. It is a part of the Standard Conjectures on algebraic cycles that these classes can be lifted to mutually orthogonal idempotents π_i, such that

$$\sum_{i=1}^{2n} \pi_i = \Delta$$

in $CH^n(X \times X)$. In [27] Murre conjectured that, moreover, the correspondences $\pi_0, \ldots, \pi_{j-1}$ and $\pi_{2j+1}, \ldots, \pi_{2n}$ act as zero on $CH^j(X)_{\mathbb{Q}}$, for any $0 \le j \le n$, the decreasing filtration

$$F^i CH^j(X)_{\mathbb{Q}} = \ker(\pi_{2j_*}) \cap \ker(\pi_{2j-1_*}) \cap \ldots \cap \ker(\pi_{2j-i+1_*})$$

independent of the choice of $\pi_0, \ldots, \pi_{2n}$, and

$$F^1 CH^j(X)_{\mathbb{Q}} = CH^j(X)_{\mathrm{hom},\mathbb{Q}}$$

for each $0 \le j \le n$.

Murre's conjectures are equivalent to the conjectures of Beilinson and Bloch, taken for all smooth and projective X over k, see [20]. For short, we will write

$$M^i(X) = (X, \pi_i, 0) ,$$

so that $M(X)$ is the direct sum of the motives $M^i(X)$ for all $i = 0, \ldots, 2n$.

If P_0 is a k-rational point on X, then

$$\pi_0 = [P_0 \times X] ,$$
$$\pi_{2n} = [X \times P_0]$$

and

$$M^0(X) = \mathbb{1} ,$$
$$M^{2n}(X) = \mathbb{L}^n$$

in $\mathsf{C}(k)$.

If C is a smooth projective curve, then π_1 is a difference between Δ and the sum of π_0 and π_2, and we obtain the well-known decomposition

$$M(C) = \mathbb{1} \oplus M^1(C) \oplus \mathbb{L} \tag{1}$$

in $\mathsf{C}(k)$. Murre's conjectures are true for curves. The motives $\mathbb{1}$ and $\mathbb{L}$ are evenly 1-dimensional, and the motive $M^1(C)$ is oddly $2g$-dimensional, where g is the genus of the curve C, see [22].

If $n > 1$, one can construct the Picard and its dual Albanese projector, π_1 and π_{2n-1}, which determine the Picard motive $M^1(X)$ and the Albanese motive $M^{2n-1}(X)$ respectively, both with coefficients in $\mathbb{Q}$, which have the expected behaviour, see the details in [26].

Let S be a smooth projective surface having a k-rational point P_0 on it. Subtracting, π_0, π_4, the Picard and Albanese projectors π_1 and π_3 from the diagonal Δ_S we get the middle projector π^2. Respectively, we obtain the decomposition of $M(S)$ into the direct sum of five motives $M^i(S)$, $i = 0, \ldots, 4$, in the category

$\mathsf{C}(k)_{\mathbb{Q}}$. The latter decomposition can be refined further by splitting the algebraic part from $M^2(S)$, see [21]. Namely, let ρ be the Picard number of S and choose ρ divisors $D_1, \ldots, D_\rho$ whose cohomology classes generate the second Weil cohomology group $H^2(S)$. Choose the Poincaré dual divisors $D'_1, \ldots, D'_\rho$, so that the intersection number $\langle D_i \cdot D'_j \rangle$ is the Kronecker symbol. For each index i let $\pi_{2,i}$ be class of the product $D_i \times D'_i$. Then π_2 decomposes into the algebraic idempotent π_2^{alg}, i.e. the sum of projectors $\pi_{2,1}, \ldots, \pi_{2,\rho}$, and the transcendental projector π_2^{tr}, i.e. the difference between π_2 and $\pi_{2,\mathrm{alg}}$. The resulting decomposition is

$$M(S) = \mathbb{1} \oplus M^1(S) \oplus \mathbb{L}^{\oplus \rho} \oplus M^2_{\mathrm{tr}}(S) \oplus M^3(S) \oplus \mathbb{L}^2 \tag{2}$$

in $\mathsf{C}(k)_{\mathbb{Q}}$. The Murre conjectures are known to be true for surfaces, except for independence of the filtration on the choice of the projectors π_i, and the latter is true if the motive $M(S)$ is finite-dimensional. If the surface S is regular, then $M^1 = M^3 = 0$.

In dimension 3 some partial results are obtained too. In [27] Murre studied the case $X = S \times C$, where S is a surface and C is a curve. The motive of a smooth projective Fano threefold is finite-dimensional and the explicit Chow-Künneth decomposition of such a motive is studied in [16].

Let now X be a smooth hypersurface in $\mathbb{P}^{n+1}$. The dimension of $H^j(X)$ is 0 if if j is odd and $j \neq n$, and it is 1 if j is even and $j \neq n$. Let b_n be the dimension of $H^n(X)$. Then all cohomology groups $H^{2j}(X)$ are algebraic, for $j \neq n$. Let Y be a general hyperplane section of X, and let γ be its class in $CH^1(X)$. For any number j between 0 and n let γ^j be the j-fold self-intersection of the class γ in $CH^j(X)$. By the Lefschetz hyperplane section theorem, the vector space $H^{2j}(X)$ is generated by the cycle class γ^j, if $2j \neq n$. For any integer $0 \leq i \leq 2n$ let

$$\pi_i = \begin{cases} 0 & \text{if } i = 2j+1,\ 0 \leq j \leq n-1 \text{ and } i \neq n \\ \frac{1}{\deg(X)} \cdot \gamma^{n-j} \times \gamma^j & \text{if } i = 2j,\ 0 \leq j \leq n \text{ and } i \neq n \end{cases}$$

and let

$$\pi_n = \Delta_X - \sum_{\substack{i=0 \\ i \neq n}}^{2n} \pi_i \ .$$

Such defined correspondences $\pi_0, \ldots, \pi_{2n}$ give the Chow-Künneth decomposition of the diagonal for X, but it is not clear whether they fully satisfy the Murre conjectures.

3. Essential (in)decomposability of motives

Let k be an arbitrary field. For any field extension L/k and any non-negative integer m let

$$\mathbf{t}^m CH^p(X_L)$$

be the subgroup in $CH^p(X_L)$ generated by the images of all pullback homomorphisms from $CH^p(X_K)$ to $CH^p(X_L)$ induced by field embeddings $K \hookrightarrow L$ over k with

$$\mathrm{tr.deg}(K/k) \leq m \ .$$

For convenience, let also

$$\mathbf{t}^{-1}CH^p(X_L) = 0 \ .$$

Then we get an increasing filtration on $CH^p(X_L)$, such that

$$\mathbf{t}^p CH^p(X_L) = CH^p(X_L)$$

and

$$\mathbf{t}^m CH^p(X_L) = 0$$

if $m > p$. We also have the graded components

$$\mathrm{Gr}^m_{\mathbf{t}} CH^p(X_L) = \mathbf{t}^m CH^p(X_L)/\mathbf{t}^{m-1}CH^p(X_L)$$

associated to $\mathbf{t}$.

The transcendental filtration $\mathbf{t}$ induces the filtration on the groups $A^p(X_L)$, and we have the corresponding graded pieces. If, moreover, k is a subfield in $\mathbb{C}$, and L is a field extension of k embeddable into $\mathbb{C}$ over k, the filtration $\mathbf{t}$ induces the filtrations on the Abel-Jacobi kernels $T^p(X_L)$, as defined in [17].

The action of correspondences preserves the transcendental filtration on Chow groups and induces the action on the corresponding graded pieces. For short, let

$$c_0(X) = \mathrm{Gr}^n_{\mathbf{t}} CH_0(X_{k(X)}) \ ,$$

$$a_0(X) = \mathrm{Gr}^n_{\mathbf{t}} A_0(X_{k(X)}) \ .$$

and

$$t_0(X) = \mathrm{Gr}^n_{\mathbf{t}} T_0(X_{k(X)}) \ ,$$

where n is the dimension of X and

$$T_i(X_L) = T^{n-i}(X_L)$$

for any i. That is, $c_0(X)$ is the Chow group 0-cycles on the product of X and $\mathrm{Spec}(k(X))$ over $\mathrm{Spec}(k)$ modulo cycle classes whose transcendental level is strictly smaller than the dimension of X, and similarly for $a_0(X)$ and $t_0(X)$.

If $\mathbf{t}^{n-1}CH^n(X_{k(X)})$ contains a degree 1 class, the inclusion of $A^n(X_{k(X)})$ into $CH^n(X_{k(X)})$ induces an isomorphism between $a_0(X)$ and $c_0(X)$. Indeed, since

$$\mathbf{t}^{n-1}A^n(X_{k(X)}) = \mathbf{t}^{n-1}CH^n(X_{k(X)}) \cap A^n(X_{k(X)})$$

by definition, the homomorphism from $a_0(X)$ to $c_0(X)$ is injective. Let Z_1 be a degree 1 cycle whose class is in $\mathbf{t}^{n-1}CH^n(X_{k(X)})$. Then any cycle class α in $CH^n(X_{k(X)})$ is congruent to the cycle class

$$\alpha - \deg(\alpha) \cdot [Z_1]$$

of degree 0 modulo $\mathbf{t}^{n-1}CH^n(X_{k(X)})$.

Let η be the generic point of X. The canonical morphism from η to X induces the pullback homomorphism

$$CH^n(X \times X) \to CH^n(X_{k(X)}) \ ,$$

which computes the value $\Phi(\eta)$ of a correspondence

$$\Phi \in CH^n(X \times X)$$

at the generic point η. For any two cycle classes ϕ and ψ in $CH^n(X_{k(X)})$, let Φ and Ψ be their spreads as codimension n cycle classes on $X \times X$. Define the product of ϕ and ψ by the formula

$$\phi \bullet \psi = (\Phi \circ \Psi)(\eta) \ ,$$

see [21]. The value $\Delta(\eta)$, i.e. the generic 0-cycle on $X_{k(X)}$, is the unit for this product, which will be denoted by $\mathbf{1}$.

When $\mathbf{t}^{n-1}CH^n(X_{k(X)})$ contains a degree 1 cycle class, one can transfer the bullet product from $c_0(C)$ to $a_0(X)$. Namely, for any two cycle classes α and β in $a_0(X)$, the bullet product of α and β in $a_0(X)$ is the difference between $\alpha \bullet \beta$ and $\deg(\alpha \bullet \beta) \cdot [Z_1]$ in $c_0(X)$, where Z_1 is a degree 1 cycle. The unit $\mathbf{1}$ in $a_0(X)$ is represented by the degree 0 zero-cycle $P_\eta - Z_1$. If $X(k) \neq \emptyset$, then Z_1 can be chosen to be a point $P_0 \in X(k)$. Then

$$\mathbf{1} = [P_\eta - P_0]$$

in $a_0(X)$.

Let Y be another smooth projective connected variety over k. The above homomorphism has the obvious generalization,

$$CH^n(Y \times X) \to CH^n(X_{k(Y)}) \ ,$$

which computes the value $\Phi(\xi)$ of a correspondence $\Phi \in CH^n(Y \times X)$ an the generic point ξ of the variety Y.

Assume that Y is of the same dimension n. A cycle class of codimension n on $Y \times X$ is said to be balanced from the left (right) if it can be represented by an algebraic cycle supported on closed subschemes of type $V \times X$ (of type $X \times V$), where V is a closed subscheme of positive codimension in X. Let

$$BCH^n(Y \times X)$$

be the subgroup of balanced correspondences in $CH^n(Y \times X)$, i.e. the subgroup generated by cycles classes balanced from the left or right on $Y \times X$.

The notion of a balanced correspondence descends from the work of Bloch, [11], Bloch and Srinivas, [12], and is straightforwardly connected to the notion of a generic zero-cycle[2]. The homomorphism computing the values of correspondences at the generic point induces an isomorphism

$$\frac{CH^n(Y \times X)}{BCH^n(Y \times X)} \stackrel{\sim}{\to} \frac{CH^n(X_{k(Y)})}{\mathbf{t}^{n-1}CH^n(X_{k(Y)})} \ ,$$

which is a straightforward generalization of Lemma 4.7 in [21]. When $Y = X$, it gives an isomorphism

$$\frac{CH^n(X \times X)}{BCH^n(X \times X)} \stackrel{\sim}{\to} c_0(X) \ , \tag{3}$$

which allows us to identify $c_0(X)$ with the quotient of the ring of correspondences $CH^n(X \times X)$ by the ideal of balanced classes $BCH^n(X \times X)$.

[2]In Appendix to Lecture 1 in [11] Spencer Bloch mentioned that "The idea that one could deduce interesting information about the Chow group by considering the generic zero-cycle was suggested by Colliot-Thélène. I am indebted to him for letting me steal it".

Warning 2. One can also introduce the balanced subgroups in $A^n(Y \times X)$, and then a temptation would be to describe $a_0(X)$ factoring balanced cycle classes in $A^n(X \times X)$. This does not work as the pullback homomorphism from $A^n(X \times X)$ to $A^n(X_{k(X)})$ is not in general surjective.

Definition 3. We will say that a correspondence Σ from Y to X is *essential* if it is not torsion, not balanced and not numerically trivial on $Y \times X$. If the diagonal class Δ on X can be represented as a sum of two essential correspondences,

$$\Delta = \Lambda + \Xi \ ,$$

then Δ is *integrally essentially decomposable*. Otherwise, Δ is *integrally essentially indecomposable*. If Δ is essentially decomposable and, moreover, Λ and Ξ are orthogonal idempotents in $CH^n(X \times X)$, then we will say that the motive $M(X)$ is *integrally essentially decomposable*. Otherwise, $M(X)$ is *integrally essentially indecomposable*.

Throughout, we will use the following rule of notation: if Λ, Ξ, Σ,... are elements in $CH^n(X \times X)$, then let λ, ξ, σ, ... are their classes modulo balanced cycles on $X \times X$, i.e. the classes in $c_0(X)$. In particular, $\mathbf{1}$ is the class δ of Δ modulo balanced cycles. If Δ is balanced, then $\mathbf{1} = 0$ and $c_0(X)$ vanishes. Definition 3 can be re-stated in terms of $c_0(X)$.

Definition 4. The Chow group $CH_0(X)$ is said to be *integrally essentially decomposable*, if $\mathbf{1}$ is a sum of two orthogonal non-torsion idempotents in $c_0(X)$. If no such a decomposition is possible, then $CH_0(X)$ is *integrally essentially indecomposable.* In other words, $CH_0(X)$ decomposes essentially, if the ring $c_0(X)$ is decomposable into two direct summands as a module over itself, and these summands are non-torsion.

Warning 5. If $M(X)$ is integrally essentially decomposable, then so is the group $CH_0(X)$. The converse assertion is, in general, not true, as the cycle classes in the ideal $BCH^n(X \times X)$ can be not nilpotent and hence idempotents can be not liftable from $c_0(X)$ to $CH^n(X \times X)$.

Remark 6. Definitions 3 and 4 can be also given for Chow groups in coefficients in $\mathbb{Q}$, or in any ring R of characteristic 0. Then the following rule applies. If $M(X)$, as an object of $\mathsf{C}(k)_{\mathbb{Q}}$, or $CH^0(X)_{\mathbb{Q}}$ is integrally essentially decomposable, they essentially decompose rationally. If they are essentially indecomposable rationally, a fortiori they are essentially indecomposable integrally.

Remark 7. Definitions 3 and 4 can be certainly given for any adequate equivalence relation on algebraic cycles. In particular, we have the notion of essential (in)decomposability of the diagonal class and the motive $\bar{M}(X)$ modulo numerical equivalence relation.

Taking into account the isomorphism (3), one can think of $c_0(X)$ as the *essential* Chow group of 0-cycles modulo rational equivalence on X. The essential decomposability property of $CH_0(X)$, or, equivalently, the decomposability property of $c_0(X)$, is a birational invariant of X.

Let, for example, C_1 and C_2 be two smooth projective curves both having a rational point over k, and let J_1 and J_2 be their Jacobians. The composition of the obvious homomorphisms

$$\frac{CH^1(C_1 \times C_2)}{BCH^1(C_1 \times C_2)} \to \mathrm{Hom}_{\mathsf{C}(k)}(M^1(C_1), M^1(C_2)) \tag{4}$$

and

$$\mathrm{Hom}_{\mathsf{C}(k)}(M^1(C_1), M^1(C_2)) \to \mathrm{Hom}(J_1, J_2)\ , \tag{5}$$

is an isomorphism by Theorem 11.5.1 in [8]. It follows that both homomorphisms are isomorphisms too.

If $C_1 = C_2 = C$, the isomorphisms (4) and (5) bring information about the structure of the motive $M(C)$. The classical fact is that $M(C)$ is essentially indecomposable. In terms of the decomposition (1), it means that the middle motive $M^1(C)$ is integrally indecomposable, i.e. indecomposable in the category $\mathsf{C}(k)$. Indeed, the Jacobian J of the curve C is a simple principally polarized abelian variety, so that the ring $\mathrm{End}(J)$ has no nonzero orthogonal idempotents whose sum would be id_J. Since $\mathrm{End}(J)$ is isomorphic to $\mathrm{End}(M^1(C))$, the latter ring possesses the same property.

Now let us also look at the notion of integral essential (in)decomposability in dimension 2. Let S be a smooth projective connected surface over a field k. Recall that the motive $M(S)$ decomposes in the standard Chow-Künneth way, as given by the formula (2). If $M(S)$ is essentially decomposable, the corresponding integral decomposition of the diagonal induces the decomposition of the transcendental projector $\pi^2_{\mathrm{tr}}(S)$ and, accordingly, the decomposition of the transcendental motive $M^2_{\mathrm{tr}}(S)$ into two nonzero direct summands in $\mathsf{C}(k)_{\mathbb{Q}}$. Since such a decomposition comes from integral projectors modulo balanced cycles, one can say that essential decomposition of $M(S)$ gives a hint what should be considered as an integral decomposition of the motive $M^2_{\mathrm{tr}}(S)$.

To be a bit more precise, we consider a homomorphism

$$CH^2(S \times S) \to \mathrm{End}_{\mathsf{C}(k)_{\mathbb{Q}}}(M^2_{\mathrm{tr}}(S))$$

sending any correspondence

$$\Sigma \in CH^2(S \times S)$$

to the endomorphism

$$\Sigma_{\mathrm{tr}} = \pi^2_{\mathrm{tr}}(S) \circ \Sigma \circ \pi^2_{\mathrm{tr}}(S)\ .$$

Clearly, it factorizes through the homomorphism

$$c_0(S) \to \mathrm{End}_{\mathsf{C}(k)_{\mathbb{Q}}}(M^2_{\mathrm{tr}}(S))\ , \tag{6}$$

sending $\sigma = [\Sigma]$ to

$$\sigma_{\mathrm{tr}} = [\Sigma_{\mathrm{tr}}]\ .$$

Localizing $c_0(S)$ with $\mathbb{Q}$, the latter homomorphism becomes an isomorphism by Theorem 4.3 in [21]. Its inverse acts as follows. Take an endomorphism Σ_{tr} of the motive $M^2_{\mathrm{tr}}(S)$ and restrict it on $U \times S$, where U is a Zariski open subset in S. Such restrictions are compatible, when U runs through all Zariski open subsets in S, which gives the cycle class $\Sigma_{\mathrm{tr}}(\eta)$ on $S_{k(S)}$, where η is the generic point of the surface S. In other words, the inverse isomorphism computes the value of Σ_{tr} at the generic point η.

Definition 8. We will say that the transcendental motive $M^2_{\mathrm{tr}}(S)$ *decomposes integrally*, if the entire motive $M(S)$ decomposes essentially. If at that the diagonal class Δ of the surface S decomposes into a sum of two essential integral orthogonal idempotents Λ and Ξ, we take their classes λ and ξ in $c_0(X)$, and apply the homomorphism (6) above. Then we obtain two orthogonal idempotents λ_{tr} and ξ_{tr} splitting the transcendental motive $M^2_{\mathrm{tr}}(S)$ into two nontrivial components. Although these idempotents are born with coefficients in $\mathbb{Q}$, the fact that they come from $c_0(S)$ allows us to look at the corresponding decomposition as an integral decomposition of $M^2_{\mathrm{tr}}(S)$. If the transcendental motive $M^2_{\mathrm{tr}}(S)$ is not integrally decomposable, then we will naturally say that it is *integrally indecomposable*.

Remark 9. According to Definition 8, integral (in)decomposability of the transcendental motive $M^2_{\mathrm{tr}}(S)$ is the same as essential (in)decomposability of the entire motive $M(S)$, in case when we deal with smooth projective surfaces over the ground field. However, this extra piece of terminology can be useful in making analogies between the conjectural integral indecomposability of the transcendental motive $M^2_{\mathrm{tr}}(S)$, and the integral indecomposability of the transcendental Hodge structure of S, which is, in general, known to be false, see [2]. If $M(S)$ is essentially decomposable, which is equivalent to saying that $M^2_{\mathrm{tr}}(S)$ decomposes integrally, then $CH_0(S)$ is essentially decomposable. By negating this implication, if $CH_0(S)$ is essentially indecomposable, then $M(S)$ is essentially indecomposable, i.e. $M^2_{\mathrm{tr}}(S)$ is integrally indecomposable.

Remark 10. Let A be an abelian group, and let α be an element in $A_{\mathbb{Q}}$. We will say that the element α is *integral* if it is in the image of the canonical homomorphism from A to $A_{\mathbb{Q}}$. In this terminology, $M^2_{\mathrm{tr}}(S)$ decomposes integrally, if it decomposes into two nontrivial summands and the corresponding idempotents are integral modulo balanced cycle classes in $CH^2(S \times S)_{\mathbb{Q}}$.

Remark 11. Definition 8 can be given with regard to any adequate equivalence relation on algebraic cycles. In particular, we have the notion of integral (in)decomposability of the motive $\bar{M}^2_{\mathrm{tr}}(S)$ in the category $\mathsf{N}(k)_{\mathbb{Q}}$ and the same logic modulo numerical equivalence as in Remark 9.

Remark 12. If $M^2_{\mathrm{tr}}(S)$ is integrally decomposable, then it decomposes rationally. By negation, if $M^2_{\mathrm{tr}}(S)$ is rationally indecomposable, then it is integrally indecomposable. We will use this observation in Propositions 14 and 15 below.

Lemma 13. *Let L be a field extension over k. If $M(S_L)$ is integrally essentially indecomposable, then $M(S)$ is integrally essentially indecomposable. In transcendental terms, if $M^2_{\mathrm{tr}}(S_L)$ is integrally indecomposable, then $M^2_{\mathrm{tr}}(S)$ is integrally indecomposable.*

Proof. Suppose that the motive $M(S_L)$ is essentially indecomposable, but the motive $M(S)$ is essentially decomposable. Then $M(S)$ splits into two nontrivial direct summands, say M and N, in the category $\mathsf{C}(k)_{\mathbb{Q}}$, and the corresponding projectors p and q are integral. Extending scalars from k to L, we obtain the decomposition of $M(S_L)$ into the motives M_L and N_L by means of the integral projectors p_L and q_L on the surface S_L over L. Since the motive $M^2_{\rm tr}(S_L)$ is integrally indecomposable, it follows that either p_L or q_L is zero. If, say, $p_L = 0$, then p must be nilpotent by the main result in [15]. Then $M = 0$, which is a contradiction, as M is nontrivial. □

Now we have to show that surfaces with integrally indecomposable $M^2_{\rm tr}(S)$ exist. If C is a smooth projective curve over k with $C(k) \neq \emptyset$, then the motive $M^2_{\rm tr}(C \times \mathbb{P}^1)$ is integrally indecomposable, as it trivial. The first nontrivial examples of integrally indecomposable transcendental motives are provided by the following two propositions.

Proposition 14. *Let S be an abelian surface isogenous to the self-product of an elliptic curve with complex multiplication over k. Then $M^2_{\rm tr}(S)$ is rationally and, hence, integrally indecomposable.*

Proof. The surface S is ρ-maximal by Proposition 3 in [7]. Therefore, $\rho(S) = 4$ and hence $\dim(M^2_{\rm tr}(S)) = 2$. Suppose $M^2_{\rm tr}(S)$ is integrally decomposable into two submotives, say M and N. As the dimension of $M^2_{\rm tr}(S)$ is 2, the dimension of M and N is 1. Applying Proposition 10.3 in [22], we see that M must be isomorphic to the Lefschetz motive $\mathbb{L}$, and the same for N. It follows that the Picard number of S is 6. This is a contradiction. □

Proposition 15. *Let S be an algebraic $K3$-surface over k, and assume that its motive $M(S)$ is finite-dimensional. Then $M^2_{\rm tr}(S)$ is rationally and, therefore, integrally indecomposable.*

Proof. Suppose $M^2_{\rm tr}(S)$ is integrally decomposable. Even more so, it is rationally decomposable. Passing to Hodge structures via Hodge realization, we see that the rational transcendental Hodge structure of S decomposes into two nontrivial components. Since finite-dimensional motives do not contain homologically phantom submotives by Proposition 7.5 in [22], the components in the rational transcendental Hodge structure of S are nontrivial. This contradicts to the main result in [38]. □

Example 16. Let $(x : y : z : t)$ be homogeneous coordinates in $\mathbb{P}^3$. A hypersurface S of degree d in $\mathbb{P}^3$ is said to be of Weil type, if S can be given by the equation

$$f(x, y) + g(z, t) = 0 ,$$

where f and g are two forms of the degree d over the ground field. For example, the Fermat hypersurface of degree d in $\mathbb{P}^3$ is of Weil type. We will also say that S is of Shioda type, if it is given by the equation

$$xy^{d-1} + yz^{d-1} + zx^{d-1} + t^d = 0$$

whose coefficients lie in $\mathbb{Q}$. The motives of Weil hypersurfaces are finite-dimensional. That can be deduced from the results in [30]. It is also easy to construct a dominant rational map from the degree d Fermat hypersurface onto the Shioda hypersurface of the same degree, see [31]. Therefore, the motive of the Shioda hypersurface in $\mathbb{P}^3$ is finite-dimensional too. Therefore, $M^2_{\rm tr}(S)$ is integrally indecomposable, if S is a $K3$ hypersurface of Weil or Shioda type. Certainly, if S is the resolution of double points on the Kummer quartic in $\mathbb{P}^3$, then the motive $M^2_{\rm tr}(S)$ is finite-dimensional and hence integrally indecomposable.

4. The self-product of a curve mapped onto a CM elliptic curve

In all the examples considered above, the integral indecoposability of the transcendental motive $M^2_{\rm tr}(S)$ is a consequence of its rational indecoposability. The aim of this section is to show an example of a surface, whose transcendental motive decomposes rationally, but it is integrally indecomposable.

Let C be a smooth projective curve over a field k, and assume that $C(k) \neq \emptyset$. The purpose of this section is to show that the motive $M^2(C \times C)$ is essentially indecomposable, provided C has enough morphisms onto an elliptic curve with complex multiplication.

Let

$$p_{1256} : C \times C \times C \times C \times C \times C \to C \times C \times C \times C$$

be the projection onto the product of the first, second, fifth and sixth factors, and let

$$\mathrm{id} \times \Delta \times \Delta \times \mathrm{id} : C \times C \times C \times C \to C \times C \times C \times C \times C \times C \;,$$

be the closed imbedding induced by the diagonal embedding of the second factor into the product of the second and third factors, and the diagonal embedding of the third factor into the product of the fourth and fifth factors. These two morphisms induce two pullback homomorphisms

$$p^*_{1256} : CH^2(C \times C \times C \times C) \to CH^2(C \times C \times C \times C \times C \times C)$$

and

$$(\mathrm{id} \times \Delta \times \Delta \times \mathrm{id})^* : CH^3(C \times C \times C \times C \times C \times C) \to CH^3(C \times C \times C \times C)$$

respectively. Let Σ be a codimension 1 cycle class on $C \times C$, and let

$$i_\Sigma : CH^2(C \times C \times C \times C \times C \times C) \to CH^3(C \times C \times C \times C \times C \times C)$$

be the homomorphism of intersection with the cycle class

$$[C \times C] \times \Sigma \times [C \times C]$$

on the 6-fold product of the curve C. Let also

$$p_{14} : C \times C \times C \times C \to C \times C$$

be the projection onto the product of the first and fourth factors, and let

$$p_{14*} : CH^3(C \times C \times C \times C) \to CH^1(C \times C)$$

be the induced pushforward homomorphism on Chow groups. Define the *convolution* by Σ homomorphism

$$cv^0_\Sigma : CH^2(C \times C \times C \times C) \to CH^1(C \times C)$$

to be the composition

$$p_{14*} \circ (\mathrm{id} \times \Delta \times \Delta \times \mathrm{id})^* \circ i_\Sigma \circ p_{1256}^* \ .$$

For example, if A and B are two cycle classes in $CH^1(C \times C)$, then

$$cv_\Sigma^0(\mathrm{A} \times \mathrm{B}) = \mathrm{B} \circ \Sigma \circ \mathrm{A}$$

and

$$cv_\Sigma^0(\mathrm{A} \otimes \mathrm{B}) = \mathrm{B} \circ \Sigma^{\mathrm{t}} \circ \mathrm{A} \ . \tag{7}$$

Let J be the Jacobian of the curve C. A *convolution* by Σ *augmented* by J is the composition

$$cv_\Sigma : CH^2(C \times C \times C \times C) \to \mathrm{End}(J) \ ,$$

of the convolution cv_Σ^0, the factorization of $CH^1(C \times C)$ modulo balanced cycles, and the homomorphisms (4) and (5).

Similarly, one can construct the convolutions with coefficients in $\mathbb{Q}$.

Let E be an elliptic curve over k and let

$$f : C \to E$$

be a nonconstant regular morphism of degree

$$n = \deg(f)$$

from C onto E over k. Then we have the correspondences

$$\Gamma_f^{\mathrm{t}} \Gamma_f \in CH^1(C \times C)$$

and

$$(\Gamma_f^{\mathrm{t}} \circ \Gamma_f) \otimes (\Gamma_f^{\mathrm{t}} \circ \Gamma_f) = (\Gamma_f^{\mathrm{t}} \otimes \Gamma_f^{\mathrm{t}}) \circ (\Gamma_f \otimes \Gamma_f) \in CH^2((C \times C) \times (C \times C)) \ .$$

Respectively, we also have the idempotent

$$\frac{1}{n} \cdot \Gamma_f^{\mathrm{t}} \Gamma_f \ ,$$

splitting $M(E)$ from $M(C)$, and the idempotent

$$\frac{1}{n} \cdot \Gamma_f^{\mathrm{t}} \Gamma_f \otimes \Gamma_f^{\mathrm{t}} \Gamma_f \ ,$$

splitting $M(E \times E)$ from $M(C \times C)$ in $\mathsf{C}(k)_{\mathbb{Q}}$.

Identify the Jacobian of E with E via the neutral element O in a chosen group law on E. The morphism f induces the morphisms

$$f^* : E \to J \qquad \text{and} \qquad f_* : J \to E \ ,$$

such that $f_* f^* = n$. Let

$$e_f^0 = f^* f_* \ ,$$

and let

$$e_f = \frac{1}{n} \cdot e_f^0$$

be the idempotent which induces the splitting of E from J in the category of abelian varieties up to isogeny, see Section 5.3 in [8].

It is not hard to see that

$$cv_{\Delta}\left(\frac{1}{n^2}\cdot\Gamma_f^{\rm t}\Gamma_f\otimes\Gamma_f^{\rm t}\Gamma_f\right)=e_f$$

in $\mathrm{End}_{\mathbb{Q}}(J)$.

Let g be the genus of C, let G be a finite group of automorphisms of the curve C, and let

$$V_1,\dots,V_r$$

be the irreducible representations of the G-module

$$H^0(\Omega_C)\ ,$$

where Ω_C is the sheaf of regular 1-forms on the curve C. Assume there exists an elliptic curve E with complex multiplication over k, and non-constant regular morphisms

$$\phi_i : C\to E\ ,$$

for each index i, such that the image of the pullback homomorphism

$$\phi_i^* : H^0(\Omega_E)\to H^0(\Omega_C)\ ,$$

is a subgroup in V_i. In such a situation, the Jacobian J of the curve C is isogenous to the self-product E^g of g copies of the curve E, and the surface $C\times C$ is ρ-maximal, see Lemma 2 and Proposition 5 in [7]. Therefore, if C enjoys the assumption above, we will say that C is a curve with *elliptically split* Jacobian. If, moreover, $g>1$, the degree of each morphism ϕ_i is greater than 1, and, therefore, J is isogenous but not regularly isomorphic to E^g.

So, since now, we will assume that C is a curve with elliptically split Jacobian. In such a case the Neron-Severi group $NS(C\times C)$ can be computed by the formula

$$NS(C\times C)=\mathbb{Z}\oplus\mathbb{Z}\oplus\mathrm{Hom}(J,J)\ ,$$

and since E is an elliptic curve with complex multiplication over k and J is isogenous to E^g, the rank of the abelian group $\mathrm{Hom}(J,J)$ is equal to $2g^2$, see page 104 in loc.cit. The second Betti number for the surface $E\times C$ is $4g+2$ and the Picard number is $2g+2$ by Lemma 1 in [7]. Hence,

$$\dim(M^2_{\rm tr}(E\times C))=2g\ ,$$

and, similarly,

$$\dim(M^2_{\rm tr}(C\times C))=2g^2\ .$$

Let

$$\tau\in H^0(\Omega_E)$$

be a generator in the one-dimensional space of global sections of the sheaf of regular 1-forms on E. For each index i choose a subset G_i in G, such that

$$\zeta^*\phi_i^*(\tau)\ ,\quad \zeta\in G_i\ ,$$

form a basis in V_i. Let

$$f : C\to E^g$$

be a regular morphism constructed by the morphisms $\phi_i\zeta$, where $i \in \{1, \dots, r\}$ and $\zeta \in G_i$, as in the proof of Lemma 2 in [7], and let

$$f_i : C \to E$$

be the composition of f with the i-th projection from E^g onto the i-th factor E. Let also

$$n_i = \deg(f_i) \ .$$

Now we have exactly g regular morphisms

$$f_1, \dots, f_g$$

from C onto E, each of which is a composition of ϕ_i and $\zeta \in G_i$.

Let

$$I = \{1, \dots, g\} \ .$$

For each index $i \in I$ we now have the idempotent

$$e_i = \frac{1}{n_i} \cdot e_i^0 \ ,$$

where

$$e_i^0 = f_i^* f_{i*} \ ,$$

see Section 5.3 in [8]. If

$$E_i = \mathrm{im}(f_i)$$

is the image of f_i inside the Jacobian J, then e_i is the uniquely defined symmetric idempotent in $\mathrm{End}_{\mathbb{Q}}(J)$ corresponding to the elliptic curve E_i inside J, see Theorem 5.3.2 in [8], and the integral endomorphism e_i^0 is the norm-endomorphism of the curve E_i in J.

For short, let

$$\Theta = \pi^2_{\mathrm{tr}}(E \times E)$$

be the transcendental projector on the product elliptic surface $E \times E$. Since

$$\pi^2(E \times E) = \pi^2(E) \otimes \pi^0(E) + \pi^1(E) \otimes \pi^1(E) + \pi^0(E) \otimes \pi^2(E)$$

and

$$\dim(M^2_{\mathrm{alg}}(E \times E)) = 4 \ ,$$

one can choose two divisors D_1 and D_2, and their Poincaré dual divisors D_1' and D_2' on $E \times E$, such that, if

$$\mathrm{A}^1 = D_1 \times D_1' \ , \qquad \mathrm{A}^2 = D_2 \times D_2'$$

and

$$\mathrm{A} = \mathrm{A}^1 + \mathrm{A}^2 \ ,$$

then

$$\pi^1(E) \otimes \pi^1(E) = \mathrm{A} + \Theta \ .$$

Let also

$$\Gamma_i = \Gamma_{f_i} \ ,$$

$$\Gamma_{ij} = \Gamma_i \otimes \Gamma_j \ .$$

$$\Theta_{ij} = \frac{1}{n_i n_j} \cdot \Gamma_{ij}^{\mathrm{t}} \circ \Theta \circ \Gamma_{ij} \ ,$$

$$\mathrm{A}^1_{ij} = \frac{1}{n_i n_j} \cdot \Gamma^{\mathrm{t}}_{ij} \circ \mathrm{A}^1 \circ \Gamma_{ij} \; ,$$

$$\mathrm{A}^2_{ij} = \frac{1}{n_i n_j} \cdot \Gamma^{\mathrm{t}}_{ij} \circ \mathrm{A}^2 \circ \Gamma_{ij}$$

and

$$\mathrm{A}_{ij} = \frac{1}{n_i n_j} \cdot \Gamma^{\mathrm{t}}_{ij} \circ \mathrm{A} \circ \Gamma_{ij} \; ,$$

so that

$$\mathrm{A}_{ij} = \mathrm{A}^1_{ij} + \mathrm{A}^2_{ij}$$

for each two indices i and j between 1 and g.

In terms of motives, let

$$T = M^2_{\mathrm{tr}}(E \times E) = (E \times E, \Theta, 0) \; ,$$

where

$$\dim(T) = 2 \; ,$$

and let

$$A = (E \times E, \mathrm{A}, 0) = \mathbb{L} \oplus \mathbb{L} \; ,$$

so that

$$M^1(E) \otimes M^1(E) = A \oplus T = \mathbb{L} \oplus \mathbb{L} \oplus T \; ,$$

and hence

$$\begin{aligned} M^2(E \times E) &= (M^2(E) \otimes M^0(E)) \oplus (M^1(E) \otimes M^1(E)) \oplus (M^0(E) \otimes M^2(E)) \\ &= \mathbb{L} \oplus A \oplus T \oplus \mathbb{L} \\ &= \mathbb{L} \oplus \mathbb{L} \oplus \mathbb{L} \oplus T \oplus \mathbb{L} \; . \end{aligned}$$

Let also

$$A^1_{ij} = (C \times C, \mathrm{A}^1_{ij}, 0) = \mathbb{L} \; , \quad A^2_{ij} = (C \times C, \mathrm{A}^2_{ij}, 0) = \mathbb{L} \; ,$$

$$A_{ij} = (C \times C, \mathrm{A}_{ij}, 0) = A^1_{ij} \oplus A^1_{ij} \qquad \text{and} \qquad T_{ij} = (C \times C, \Theta_{ij}, 0)$$

be the 2-dimensional images of the motives A and T respectively inside the middle motive $M^2(C \times C)$ under the embeddings

$$\Gamma^{\mathrm{t}}_{ij} : M(E \times E) \to M(C \times C) \; .$$

The motives T_{ij} can be viewed as indecomposable "motivic atoms" inside the transcendental motive $M^2_{\mathrm{tr}}(C \times C)$. Since the motive $M(S)$ is finite-dimensional, there are no homologically phantom submotives in $M(S)$ by Proposition 7.5 in [22]. It follows that

$$M^2_{\mathrm{tr}}(C \times C) = \oplus^g_{i,j=1} T_{ij} \; ,$$

i.e. the transcendental motive $M^2_{\mathrm{tr}}(C \times C)$ consists of exactly g^2 motives T_{ij} each of which is isomorphic to the indecomposable motive T.

The following exercises give some practicing in how the motives T_{ij} are placed inside $M^2_{\mathrm{tr}}(C \times C)$. First of all,

$$M^1(E^g) = M^1(E)^{\oplus g} \; ,$$

whence

$$M^2(E \times E^g) = \mathbb{L} \oplus (M^1(E) \otimes M^1(E))^{\oplus g} \oplus M^2(E^g) \; .$$

Since

$$M^1(E)\otimes M^1(E)=\mathbb{L}^{\oplus 2}\oplus M^2_{\mathrm{tr}}(E\times E)\ ,$$

we obtain that

$$M^2(E\times E^g)=\mathbb{L}\oplus(\mathbb{L}^{\oplus 2}\oplus M^2_{\mathrm{tr}}(E\times E))^{\oplus g}\oplus M^2(E^g)\ ,$$

i.e. there are g copies of the indecomposable 2-dimensional motive $M^2_{\mathrm{tr}}(E\times E)$ as direct summands inside the motive $M^2(E\times E^g)$. Composing the embedding of $M^2_{\mathrm{tr}}(E\times E)^{\oplus g}$ into $M(E\times E^g)$ with the morphism

$$\Delta\times\Gamma^{\mathrm{t}}_f : M(E\times E^g)\to M(E\times C)\ ,$$

we obtain a morphism

$$M^2_{\mathrm{tr}}(E\times E)^{\oplus g}\to M(E\times C)\ .$$

Precomposing the latter with the j-th canonical inclusion of $M^2_{\mathrm{tr}}(E\times E)$ into $M^2_{\mathrm{tr}}(E\times E)^{\oplus g}$, we obtain the morphism from $M^2_{\mathrm{tr}}(E\times E)$ to $M(E\times C)$ which factorizes through the transcendental motive $M^2_{\mathrm{tr}}(E\times C)$. This gives g transcendental 2-dimensional motives

$$\tilde{T}_{ij}\ ,\quad j=1,\dots,g\ ,$$

inside $M^2_{\mathrm{tr}}(E\times C)$, for each fixed i.

Further we compute

$$M^2(E^g\times C)=M^2(E^g)\oplus(M^1(E)\otimes M^1(C))^{\oplus g}\oplus\mathbb{L}\ ,$$

and since

$$M^1(E)\otimes M^1(C)=\mathbb{L}^{\oplus 2g}\oplus M^2_{\mathrm{tr}}(E\times C)\ ,$$

one has g independent copies of the motive $M^2_{\mathrm{tr}}(E\times C)$ inside $M^2(E^g\times C)$. Composing the embedding of $M^2_{\mathrm{tr}}(E\times C)$ into $M^2(E^g\times C)$ with the morphism

$$\Gamma^{\mathrm{t}}_f\times\Delta : M(E^g\times C)\to M(C\times C)$$

we obtain the embedding

$$M^2_{\mathrm{tr}}(E\times C)^{\oplus g}\to M(C\times C)\ .$$

Precomposing the latter with the i-th canonical embedding of $M^2_{\mathrm{tr}}(E\times C)$ into $M^2_{\mathrm{tr}}(E\times C)^{\oplus g}$ we obtain the morphism from $M^2_{\mathrm{tr}}(E\times C)$ to $M(C\times C)$ which factorizes through the transcendental motive $M^2_{\mathrm{tr}}(C\times C)$ and, thus, gives g isomorphic copies of the motive $M^2_{\mathrm{tr}}(E\times C)$ inside $M^2_{\mathrm{tr}}(C\times C)$.

Since each $M^2_{\mathrm{tr}}(E\times C)$ consists of g transcendental 2-dimensional motives $\tilde{T}_{i1},\dots,\tilde{T}_{ig}$, we obtain that all together there are g^2 images T_{ij} of the indecomposable transcendental motive T inside $M^2_{\mathrm{tr}}(C\times C)$ under the morphisms Γ^{t}_{ij}, i.e.

$$M^2_{\mathrm{tr}}(C\times C)=\sum_{i,j=1}^{g}T_{ij}\ ,\tag{8}$$

and, in terms of projectors,

$$\pi^2_{\mathrm{tr}}(C\times C)=\sum_{i,j=1}^{g}\Theta_{ij}\ .\tag{9}$$

In the same manner,

$$M^2_{\mathrm{alg}}(C\times C)=\mathbb{L}\oplus\sum_{i,j=1}^{g}A_{ij}\oplus\mathbb{L}\ , \tag{10}$$

so that

$$\dim(M^2_{\mathrm{alg}}(C\times C))=2g^2+2\ ,$$

and, in terms of projectors,

$$\pi^2_{\mathrm{alg}}(C\times C)=\pi^2(C)\otimes\pi^0(C)+\sum_{i,j=1}^{g}\mathrm{A}_{ij}+\pi^0(C)\otimes\pi^2(C)\ .$$

Now a complete accounting of $M(C\times C)$ is this:

$$M(C\times C)=\oplus_{i=0}^{4}M^i(C\times C)\ ,$$

where

$$M^0(C\times C)=M^0(C)\otimes M^0(C)=\mathbb{1}\ ,$$

$$\begin{aligned}M^1(C\times C)&=(M^1(C)\otimes M^0(C))\oplus(M^1(C)\otimes M^0(C))\\&=M^1(C)\oplus M^1(C)\ ,\end{aligned}$$

$$\begin{aligned}M^2(C\times C)&=(M^2(C)\otimes M^0(C))\oplus(M^1(C)\otimes M^1(C))\oplus(M^0(C)\otimes M^2(C))\\&=\mathbb{L}\oplus(M^1(C)\otimes M^1(C))\oplus\mathbb{L}\\&=M^2_{\mathrm{alg}}(C\times C)\oplus M^2_{\mathrm{tr}}(C\times C)\ ,\end{aligned}$$

where $M^2_{\mathrm{tr}}(C\times C)$ and $M^2_{\mathrm{alg}}(C\times C)$ are described by (8) and (10),

$$\begin{aligned}M^3(C\times C)&=(M^2(C)\otimes M^1(C))\oplus(M^1(C)\otimes M^2(C))\\&=(\mathbb{L}\otimes M^1(C))\oplus(M^1(C)\otimes\mathbb{L})\ ,\end{aligned}$$

and

$$M^4(C\times C)=M^2(C)\otimes M^2(C)=\mathbb{L}^4\ .$$

The motives $\mathbb{L}\otimes M^1(C)$ and $M^1(C)\otimes\mathbb{L}$ are integrally indecomposable, because the Tate motive $\mathbb{T}$ is monoidally inverse to the Lefschetz motive $\mathbb{L}$.

We will also need the following notation, with regard to the structure of the motive $M(C\times C)$. Let

$$I^2=I\times I$$

be the Cartesian square of the set I. For any subset

$$U\subset I^2$$

let

$$\mathrm{A}^1_U=\sum_{(i,j)\in U}\mathrm{A}^1_{ij}\ ,$$

$$\mathrm{A}^2_U=\sum_{(i,j)\in U}\mathrm{A}^2_{ij}\ ,$$

$$\mathrm{A}_U=\sum_{(i,j)\in U}\mathrm{A}_{ij}\ ,$$

$$\Theta_U=\sum_{(i,j)\in U}\Theta_{ij}\ ,$$

be the projectors, and let

$$A^1_U = \bigoplus_{(i,j)\in U} A^1_{ij}$$

$$A^2_U = \bigoplus_{(i,j)\in U} A^2_{ij}$$

$$A_U = \bigoplus_{(i,j)\in U} A_{ij}$$

and

$$T_U = \bigoplus_{(i,j)\in U} T_{ij}$$

be the corresponding algebraic and transcendental submotives in $M(C \times C)$. If

$$W = I^2 \smallsetminus U \ ,$$

then, of course,

$$M^2_{\mathrm{alg}}(C \times C) = \mathbb{L} \oplus A_U \oplus A_W \oplus \mathbb{L} \ .$$

and

$$M^2_{\mathrm{tr}}(C \times C) = T_U \oplus T_W \ .$$

Next, let γ_i be the class of Γ_i modulo balanced cycles in $CH^1(C \times C)_{\mathbb{Q}}$. Let also γ_{ij}, α_{ij} and θ_{ij} be the classes of, respectively, the correspondences Γ_{ij}, A_{ij} and Θ_{ij} modulo balanced cycles in $CH^2((C\times C)\times(C\times C))_{\mathbb{Q}}$. The transcendental projector $\pi^2_{\mathrm{tr}}(E\times E)$ is congruent to Δ modulo balanced cycles on the self-product of the surface $E \times E$. Therefore,

$$\theta_{ij} = \frac{1}{n_i n_j} \cdot \gamma^{\mathrm{t}}_{ij}\gamma_{ij} \ ,$$

for each indices i and j. Since

$$\gamma_{ij} = \gamma_i \otimes \gamma_j \ ,$$

it follows that

$$\theta_{ij} = \frac{1}{n_i} \cdot \gamma^{\mathrm{t}}_i\gamma_i \otimes \frac{1}{n_j} \cdot \gamma^{\mathrm{t}}_j\gamma_j \ .$$

The norm-endomorphism e^0_i of the elliptic curve E_i in J can be expressed as

$$e^0_i = \gamma^{\mathrm{t}}_i\gamma_i \ ,$$

and, as we have seen above, the idempotent

$$e_i = \frac{1}{n_i} \cdot e^0_i \ ,$$

symmetric under the Rosatti involution, determines the i-th factor in E^g under the isogeny between J and E^g. The degree n_i is the exponent of the elliptic curve E_i in J. Then

$$\theta_{ij} = e_i \otimes e_j$$

in the group

$$\frac{CH^2(S \times S)_{\mathbb{Q}}}{BCH^2(S \times S)_{\mathbb{Q}}} \ .$$

Due to (9),

$$\mathbf{1} = \sum_{i,j=1}^{g} \theta_{ij}$$

in $c_0(S)_{\mathbb{Q}}$.

Lemma 17. *Let a and b be two arbitrary indices in I. In terms above,*

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\mathrm{A}_{ij}^l) = \begin{cases} -\frac{1}{2}\cdot\gamma_b^{\mathrm{t}}\gamma_a\ , & \text{if } i=a \text{ and } j=b \\ 0 & \text{otherwise} \end{cases}$$

and

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Theta_{ij}) = \begin{cases} 2\gamma_b^{\mathrm{t}}\gamma_a\ , & \text{if } i=a \text{ and } j=b \\ 0 & \text{otherwise} \end{cases}$$

for any $l=1,2$ and all i and j between 1 and g.

Proof. For any two divisors D and D' on $E\times E$,

$$\Gamma_{ij}^{\mathrm{t}}\circ(D\times D')\circ\Gamma_{ij} = (\Gamma_j^{\mathrm{t}}\circ D\circ\Gamma_i)\times(\Gamma_j^{\mathrm{t}}\circ D'\circ\Gamma_i)\ ,$$

whence

$$cv^0_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Gamma_{ij}^{\mathrm{t}}\circ(D\times D')\circ\Gamma_{ij}) = \Gamma_j^{\mathrm{t}}\circ D'\circ\Gamma_i\circ\Gamma_a^{\mathrm{t}}\circ\Gamma_b\circ\Gamma_j^{\mathrm{t}}\circ D\circ\Gamma_i\ . \tag{11}$$

If $i\neq a$, then $\Gamma_i\circ\Gamma_a^{\mathrm{t}}$ is a balanced class in the group $CH^1(E\times E)_{\mathbb{Q}}$. If $j\neq b$, then $\Gamma_b\circ\Gamma_j^{\mathrm{t}}$ is a balanced class in $CH^1(E\times E)_{\mathbb{Q}}$. In particular,

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\mathrm{A}_{ij}^l) = 0 \tag{12}$$

for $l=1,2$, if either $i\neq a$ or $j\neq b$.

For the same reason,

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Gamma_{ij}^{\mathrm{t}}\circ\Gamma_{ij}) = cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Gamma_i^{\mathrm{t}}\Gamma_i\otimes\Gamma_j^{\mathrm{t}}\Gamma_j) = \gamma_j^{\mathrm{t}}\gamma_j\gamma_b^{\mathrm{t}}\gamma_a\gamma_i^{\mathrm{t}}\gamma_i = 0\ , \tag{13}$$

if either $i\neq a$ or $j\neq b$.

Moreover, if t is different from s, then one of the two projectors $\pi^s(E)$ or $\pi^t(E)$ is a balanced cycle class on $C\times C$, whence

$$cv_{\Sigma}(\Gamma_{ij}^{\mathrm{t}}\circ(\pi^s(E)\otimes\pi^t(E))\circ\Gamma_{ij}) = 0$$

for any cycle class Σ in $CH^1(C\times C)$.

The equalities (11), (12) and (13) then give

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Theta_{ij}) = 0\ ,$$

if either $i\neq a$ or $j\neq b$.

Now assume that $i=a$ and $j=b$. In such a case,

$$cv^0_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Gamma_{ab}^{\mathrm{t}}\circ(D\times D')\circ\Gamma_{ab}) = n_a\cdot n_b\cdot\Gamma_b^{\mathrm{t}}\circ D'\circ D\circ\Gamma_a\ , \tag{14}$$

for any two divisors D and D' on $E\times E$.

Since E is an elliptic curve with complex multiplication, there is a positive integer d, not a square in $\mathbb{Z}$, such that $\mathrm{End}_{\mathbb{Q}}(E)$ is isomorphic to the imaginary quadratic field $\mathbb{Q}(\sqrt{-d})$. Let Σ be the graph of the endomorphism

$$\sqrt{-d} : E\to E\ ,$$

and consider the divisors

$$D_1 = \Delta - [O \times E] - [E \times O] \quad \text{and} \quad D_2 = \Sigma - d \cdot [O \times E] - [E \times O]$$

on $E \times E$. Since $-\frac{1}{2} \cdot D_1$ is Poincaré dual to D_1 and $-\frac{1}{2d} \cdot D_2$ is Poincaré dual to D_2, we have that

$$\mathrm{A}^1 = -\frac{1}{2} \cdot D_1 \times D_1 \qquad \text{and} \qquad \mathrm{A}^2 = -\frac{1}{2d} \cdot D_2 \times D_2 \; .$$

Then (14) gives

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\mathrm{A}^l_{ab}) = -\frac{1}{2} \cdot \gamma_b^{\mathrm{t}} \gamma_a \; ,$$

for $l = 1, 2$.

And since

$$cv^0_{\Gamma_a^{\mathrm{t}}\Gamma_b}\left(\frac{1}{n_a n_b} \cdot \Gamma^{\mathrm{t}}_{ab} \circ \Gamma_{ab}\right) = \Gamma^{\mathrm{t}}_b \circ \Gamma_a \; ,$$

it follows that

$$cv_{\Gamma_a^{\mathrm{t}}\Gamma_b}(\Theta_{ab}) = 2 \cdot \gamma_b^{\mathrm{t}} \gamma_a \; ,$$

□

For any permutation σ if the numbers $\{1, \ldots, g\}$ let

$$\sigma_{E^g} : E^g \to E^g$$

be the regular morphism permuting the factors in E^g according to the permutation σ. The morphisms $f_{i*} : J \to E$ and $f_i^* : E \to J$ induce the inverse isogenies

$$f_* : J \to E^g \qquad \text{and} \qquad f^* : E^g \to J \; .$$

Let

$$\sigma_J : J \to J$$

be the composition $f^* \circ \sigma_{E^g} \circ f_*$. Then σ_J is an element in $\mathrm{End}(J)$, which decomposes as

$$\sigma_J = \frac{1}{n_1} \cdot \gamma_1^{\mathrm{t}} \gamma_{\sigma(1)} + \ldots + \frac{1}{n_g} \cdot \gamma_g^{\mathrm{t}} \gamma_{\sigma(g)}$$

in $\mathrm{End}_{\mathbb{Q}}(J)$. Therefore, if

$$\Sigma_J = \frac{1}{n_1} \cdot \Gamma_1^{\mathrm{t}} \Gamma_{\sigma(1)} + \ldots + \frac{1}{n_g} \cdot \Gamma_g^{\mathrm{t}} \Gamma_{\sigma(g)} \; ,$$

then Σ_J is integral modulo balanced cycles on $C \times C$.

Certainly, σ_J is an automorphism, $(\sigma^{-1})_J$ is the same as $(\sigma_J)^{-1}$, and we may simply write σ_J^{-1}. If σ is the identity permutation, then Σ_J is congruent to the diagonal Δ modulo balanced cycles. Formula (7) gives that

$$cv_{\Sigma_J}(\Delta \otimes \Delta) = \sigma_J^{\mathrm{t}} \; .$$

Corollary 18. *For any permutation σ,*

$$cv_{\Sigma_J}(\mathrm{A}^l_{ij}) = \begin{cases} -\frac{1}{2n_i} \cdot \gamma^{\mathrm{t}}_{\sigma(i)} \gamma_i \ , & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}$$

and

$$cv_{\Sigma_J}(\Theta_{ij}) = \begin{cases} \frac{2}{n_i} \cdot \gamma^{\mathrm{t}}_{\sigma(i)} \gamma_i \ , & \text{if } j = \sigma(i) \\ 0 & \text{otherwise} \end{cases}$$

for any $l = 1, 2$ and all i and j between 1 and g.

Proof. This is a straightforward consequence of Lemma 17. □

For any subset K in I, let

$$e_K = \sum_{i \in K} e_i \ ,$$

and write $e_K = 0$ if K is empty. In particular,

$$e_I = \mathrm{id}_J$$

is the identity automorphism of the Jacobian J. Let also n_K be the exponent of an abelian subvariety E^K in J associated to the idempotent e_K, i.e. the minimal positive integer n_K, such that $n_K e_K$ is integral. Then we write

$$e_K = \frac{1}{n_K} \cdot e^0_K \ ,$$

where e^0_K is the norm-endomorphism of E^K, in terms of [8]. We will need the following easy lemma.

Lemma 19. *Let A and B be two subsets in I, and assume that*

$$n_K \geq 4$$

for any subset K in I, such that

$$\emptyset \neq K \neq I \ .$$

If

$$2e_A + e_B \in \mathrm{End}(J) \ ,$$

then

$$A, B \in \{\emptyset, I\} \ .$$

Proof. Let

$$S = A \cap B \ , \qquad T = A \smallsetminus B \ , \qquad R = B \smallsetminus A \ .$$

Then S, T and R are three subsets in I,

$$S \cap T = S \cap R = T \cap R = \emptyset \ ,$$

and

$$g = 2e_A + e_B = 3e_S + 2e_T + e_R$$

is integral by assumption. As

$$2e_T + 2e_R = 3g - g^2 \ .$$

is integral too, and since $T \cap R = \emptyset$, the endomorphism

$$2 \cdot e_{T \cup R}$$

is integral.

Now, if $\emptyset \neq T \cup R \neq I$, Proposition 12.1.1 in [8] gives $n_{T\cup R} = 2$, which contradicts to the assumption of the lemma.

If $T \cup R = I$, then $I = A \cup B$ and $A \cap B = \emptyset$. In such a case,

$$g = 2e_A + e_B = 2e_A + e_{I \smallsetminus A} = e_A + \mathrm{id}\ ,$$

whence e_A is integral. Therefore, either $A = \emptyset$ and then $B = I$, or $A = I$ and then $B = \emptyset$.

If $T \cup R = \emptyset$, then $A = B$, and hence $3e_A$ is integral. If $\emptyset \neq A \neq I$, Proposition 12.1.1 in [8] gives $n_A = 3$, which contradicts to the assumption of the lemma. Therefore, either $A = I$ or $\emptyset$. □

For any subset U in I^2 let

$$I_{U,\sigma} = \{i \in I \mid (i, \sigma(i)) \in U\}\ ,$$

and let

$$\sigma_U = \sum_{i \in I_{U,\sigma}} \frac{1}{n_i} \cdot \gamma_i^{\mathrm{t}} \gamma_{\sigma(i)}\ .$$

If $\sigma = 1^g$ is the identity permutation, then, for short of notation, we will write

$$I_U = I_{U,1^g}$$

and

$$e_U = e_{I_U}\ .$$

Then, of course,

$$(1^g)_U = e_U\ .$$

The endomorphisms σ_U have many nice properties. For example, one has

Corollary 20. *For any $U \subset I^2$,*

$$cv_{\Sigma_J}(\mathrm{A}_U^l) = -\frac{1}{2} \cdot \sigma_U^{\mathrm{t}}$$

for $l = 1, 2$, and

$$cv_{\Sigma_J}(\Theta_U) = 2 \cdot \sigma_U^{\mathrm{t}}$$

Proof. Straightforward from Corollary 18. □

It is also easy to see that

$$(\sigma_U)^m = (\sigma^m)_U\ ,$$

for any natural number m, so that we will simply write σ_J^m for both. If m is the order of the permutation σ, then

$$\sigma_U^m = e_U\ .$$

Another useful property of the endomorphisms σ_U is this. Let

$$I_{\sigma,U} = \{i \in I \mid (\sigma(i), i) \in U\}\ ,$$

and let

$$e_{\sigma,U} = \sum_{i \in I_{\sigma,U}} \frac{1}{n_i} \cdot \gamma_i^{\mathrm{t}} \gamma_i\ .$$

In particular,

$$e_{\mathrm{id},U} = e_U \ .$$

If m is the order of σ, it is easy to see that

$$\sigma_J \circ \sigma_U^{m-1} = e_{\sigma^{m-1},U} \ ,$$

or, equivalently,

$$\sigma_J \circ (\sigma^{-1})_U = e_{\sigma^{-1},U} \ .$$

Swapping σ and σ^{-1} yeilds

$$\sigma_J^{-1} \circ \sigma_U = e_{\sigma,U} \ ,$$

and, transposing, we obtain

$$\sigma_U^{\mathrm{t}} \circ (\sigma_J^{-1})^{\mathrm{t}} = e_{\sigma,U} \tag{15}$$

for any subset U in I^2.

The following result is Theorem A in Introduction.

Theorem 21. *Let k be a field of characteristic 0, and let C be a smooth projective curve over k. Assume that the Jacobian of C splits by an elliptic curve with complex multiplication E, i.e. there is a finite group G of automorphisms of C and non-constant regular morphisms,*

$$\phi_i : C \to E \ , \qquad i = 1, \dots, r \ ,$$

one for each irreducible representation V_i of the action of G on $H^0(\Omega_C)$, such that the image of the pullback homomorphism

$$\phi_i^* : H^0(\Omega_E) \to H^0(\Omega_C)$$

is in V_i. Assume, furthermore, that

$$\deg(\phi_i) \geq 4$$

for all i. Then the motive $M(C \times C)$ is essentially indecomposable, i.e. the transcendental motive $M^2_{\mathrm{tr}}(C \times C)$ is indecomposable integrally.

Proof. By Lemma 13, the ground field k can be algebraically closed. Assume the motive $M^2_{\mathrm{tr}}(C^2)$ decomposes integrally. According to Definition 8 and Remark 9, it means that the diagonal class of the surface C^2 decomposes into two mutually orthogonal idempotents,

$$\Delta = \Lambda + \Xi \ ,$$

in the Chow group

$$CH^2(C \times C \times C \times C) \ ,$$

such that their classes λ and, respectively, ξ modulo balanced cycles are nontrivial and non-torsion. Then, of course, we have the corresponding splitting

$$M(C \times C) = M_\Lambda \oplus M_\Xi$$

into two non-torsion motives in $\mathsf{C}(k)$.

Let g be the genus of the curve C, and let

$$I = \{1, \dots, g\} \ .$$

Construct the morphisms f_i as above, and set

$$n_i = \deg(f_i) \ ,$$

for each index i in I. Then we have the systems projectors A^1_{ij}, A^2_{ij} and Θ_{ij} on the surface $C \times C$.

For short, let

$$K = \{0, 1, 2\} \ , \quad K^2 = K \times K \ ,$$

and for each ordered pair of indices

$$(s, t) \in K^2 \smallsetminus \{1, 1\}$$

let

$$B^{s,t} = M^s(C) \otimes M^t(C) \ ,$$

and for any subset

$$L \subset K^2 \smallsetminus \{1, 1\}$$

let

$$B_L = \oplus_{(s,t) \in L} B^{s,t}$$

be the motive given by the projector

$$\mathrm{B}_L = \sum_{(s,t) \in L} \pi^s(C) \otimes \pi^t(C) \ .$$

Then

$$M(C \times C) = \oplus_{(i,j) \in I^2} (A^1_{ij} \oplus A^2_{ij} \oplus T_{ij}) \oplus (\oplus_{(s,t) \in K^2 \smallsetminus \{1,1\}} B^{s,t})$$

is the refined Chow-Künneth decomposition of $M(C \times C)$, and each direct summand in this decomposition is an indecomposable motive in $\mathsf{C}(k)_{\mathbb{Q}}$. Using the semisimplicity of the numerical category $\mathsf{N}(k)_{\mathbb{Q}}$ and Lemma 1, we obtain that there exist subsets

$$U_\Lambda \ , \ U_\Xi \ , \ V_\Lambda \ , \ V_\Xi \ , \ W_\Lambda \ , \ W_\Xi \subset \ I^2 \ ,$$

$$L_\Lambda \ , \ L_\Xi \ \subset \ K^2 \smallsetminus \{1, 1\} \ ,$$

such that

$$I^2 = U_\Lambda \cup U_\Xi = V_\Lambda \cup V_\Xi = W_\Lambda \cup W_\Xi \ ,$$

$$K^2 \smallsetminus \{1, 1\} = L_\Lambda \cup L_\Xi \ ,$$

all four unions are disjoint,

$$\bar{M}_\Lambda = \bar{A}^1_{U_\Lambda} \oplus \bar{A}^2_{V_\Lambda} \oplus \bar{T}_{W_\Lambda} \oplus \bar{B}_{L_\Lambda}$$

and

$$\bar{M}_\Xi = \bar{A}^1_{U_\Xi} \oplus \bar{A}^2_{V_\Xi} \oplus \bar{T}_{W_\Xi} \oplus \bar{B}_{L_\Xi}$$

in $\mathsf{N}(k)_{\mathbb{Q}}$. It follows that

$$\Lambda = \mathrm{A}^1_{U_\Lambda} + \mathrm{A}^2_{V_\Lambda} + \Theta_{W_\Lambda} + \mathrm{B}_{L_\Lambda} + \Upsilon_\Lambda \tag{16}$$

and

$$\Xi = \mathrm{A}^1_{U_\Xi} + \mathrm{A}^2_{V_\Xi} + \Theta_{W_\Xi} + \mathrm{B}_{L_\Xi} + \Upsilon_\Xi \tag{17}$$

for some numerically trivial correspondences Υ_Λ and Υ_Ξ in $CH^2(C \times C \times C \times C)_{\mathbb{Q}}$.

Since the motive $M(C)$ is finite-dimensional, any numerically trivial cycle class in $CH^1(C \times C)$ is nilpotent by Proposition 7.5 in [22]. On the other hand, the algebra $\mathrm{End}_{\mathbb{Q}}(J)$, being a product of fields, has no nilpotent elements in it. It follows that, for any $\Sigma \in CH^1(C \times C)$ the convolution cv_Σ takes numerically trivial correspondences in $CH^2(C \times C \times C \times C)$ to 0. In particular,

$$cv_\Sigma(\Upsilon_\Lambda) = 0 \qquad \text{and} \qquad cv_\Sigma(\Upsilon_\Xi) = 0$$

in $\mathrm{End}_{\mathbb{Q}}(J)$.

Moreover, if $(s,t) \in K^2 \smallsetminus \{1,1\}$, then at least one of the projectors, $\pi^s(C)$ or $\pi^t(C)$, is balanced on $C \times C$, so that

$$cv_\Sigma(\pi^s(C) \otimes \pi^t(C)) = \pi^t(C) \circ \Sigma^{\mathrm{t}} \circ \pi^s(C) = 0 \;,$$

whenever s is different from t. It follows that

$$cv_\Sigma(\mathrm{B}_{L_\Lambda}) = 0 \;, \qquad cv_\Sigma(\mathrm{B}_{L_\Xi}) = 0$$

in $\mathrm{End}_{\mathbb{Q}}(J)$.

Therefore, the equalities (16) and (17) yield

$$cv_\Sigma(\Lambda) = cv_\Sigma(\mathrm{A}^1_{U_\Lambda}) + cv_\Sigma(\mathrm{A}^2_{V_\Lambda}) + cv_\Sigma(\Theta_{W_\Lambda})$$

and

$$cv_\Sigma(\Xi) = cv_\Sigma(\mathrm{A}^1_{U_\Xi}) + cv_\Sigma(\mathrm{A}^2_{V_\Xi}) + cv_\Sigma(\Theta_{W_\Xi})$$

for any Σ in $CH^1(C \times C)$.

Case 1: when both sets I_{W_Λ} and I_{W_Ξ} are nonempty

By Corollary 18,

$$cv_\Delta(\Lambda) = -\frac{1}{2} \cdot e_{U_\Lambda} - \frac{1}{2} \cdot e_{V_\Lambda} + 2 \cdot e_{W_\Lambda} \;, \tag{18}$$

and

$$cv_\Delta(\Xi) = -\frac{1}{2} \cdot e_{U_\Xi} - \frac{1}{2} \cdot e_{V_\Xi} + 2 \cdot e_{W_\Xi} \;, \tag{19}$$

in $\mathrm{End}_{\mathbb{Q}}(J)$, and, since the sets I_{W_Λ} and I_{W_Ξ} are both nonempty, $e_{W_\Lambda} \neq 0$ and $e_{W_\Xi} \neq 0$.

Suppose there exists $i \in I_{U_\Lambda} \smallsetminus (I_{W_\Lambda} \cup I_{V_\Lambda})$. Multiplying (18) by e_i, we obtain

$$e_i \cdot cv_\Delta(\Lambda) = -\frac{1}{2} \cdot e_i \;.$$

Multiplying both sides by $-2n_i$, we get

$$-2 \cdot e_i^0 \cdot cv_\Delta(\Lambda) = e_i^0 \;.$$

Since $cv_\Delta(\Lambda)$ is integral and e_i^0 is the norm-endomorphism of the i-th elliptic curve inside J, the latter equality contradicts the Norm-endomorphism Criterion 5.3.4 on page 124 in [8]. Therefore, I_{U_Λ} is a subset of $I_{W_\Lambda} \cup I_{V_\Lambda}$. By symmetry, I_{V_Λ}

is a subset of $I_{W_\Lambda} \cup I_{U_\Lambda}$. Moreover, if we suppose that there exists $i \in I_{U_\Lambda} \smallsetminus I_{V_\Lambda}$, such i must be in I_{W_Λ}, and the multiplication of (18) by e_i gives

$$e_i \cdot cv_\Delta(\Lambda) = -\frac{1}{2} \cdot e_i + 2e_i \; .$$

Multiplying by $2n_i$ yields

$$2e_i^0 \cdot cv_\Delta(\Lambda) = 3e_i^0 \; ,$$

whence

$$2 \cdot (2e_i^0 \cdot cv_\Delta(\Lambda) - e_i^0) = e_i^0 \; ,$$

and we again in contradiction with the Criterion 5.3.4 in loc.cit. Therefore, $I_{U_\Lambda} \subset I_{V_\Lambda}$. By symmetry, $I_{V_\Lambda} \subset I_{U_\Lambda}$. Thus, $I_{U_\Lambda} = I_{V_\Lambda}$, and, similarly, $I_{U_\Xi} = I_{V_\Xi}$. Therefore, (18) and (19) turn into the equalities

$$cv_\Delta(\Lambda) = 2 \cdot e_{W_\Lambda} - e_{U_\Lambda} \tag{20}$$

and

$$cv_\Delta(\Xi) = 2 \cdot e_{W_\Xi} - e_{U_\Xi} \; , \tag{21}$$

respectively.

Since

$$cv_\Delta(\Delta) = \mathrm{id}$$

and hence

$$\mathrm{id} = cv_\Delta(\Lambda) + cv_\Delta(\Xi) \; ,$$

and also taking into account (20), (21), we obtain

$$\mathrm{id} = (2e_{W_\Lambda} - e_{U_\Lambda}) + (2e_{W_\Xi} - e_{U_\Xi}) \; ,$$

where the endomorphisms in the brackets are integral. Re-arranging,

$$3 \cdot \mathrm{id} = (2e_{W_\Lambda} + e_{U_\Xi}) + (2e_{W_\Xi} + e_{U_\Lambda}) \; ,$$

and the endomorphisms in the brackets are still integral. Applying Lemma 19

$$I_{W_\Lambda}, I_{W_\Xi} \in \{\emptyset, I\} \; ,$$

which contradicts to the assumption of Case 1.

Case 2: when one of the two sets I_{W_Λ} and I_{W_Ξ} is empty

If, say, I_{W_Ξ} is empty, then I_{W_Λ} must be the whole diagonal in I^2. Since the decomposition

$$\Delta = \Lambda + \Xi$$

induces a splitting of $M^2_{\mathrm{tr}}(C \times C)$ into two nontrivial components, the set W_Ξ is nonempty, however. Choose and fix an arbitrary pair

$$(i_0, j_0) \in W_\Xi \; ,$$

and let σ be a transposition of the elements i_0 and j_0 in $\{1, \ldots, g\}$. The permutation σ induces the automorphism

$$\sigma_J : J \to J \; ,$$

and the cycle class

$$\Sigma_J = \sum_{\substack{i=1 \\ i\neq i_0 \\ i\neq j_0}}^{g} \frac{1}{n_i}\Gamma_i^{\mathrm{t}}\Gamma_i + \frac{1}{n_{i_0}}\Gamma_{i_0}^{\mathrm{t}}\Gamma_{j_0} + \frac{1}{n_{j_0}}\Gamma_{j_0}^{\mathrm{t}}\Gamma_{i_0} \; .$$

By Corollary 20,

$$cv_{\Sigma_J}(\Lambda) = -\frac{1}{2}\cdot\sigma_{U_\Lambda}^{\mathrm{t}} - \frac{1}{2}\cdot\sigma_{V_\Lambda}^{\mathrm{t}} + 2\cdot\sigma_{W_\Lambda}^{\mathrm{t}} \; ,$$

and

$$cv_{\Sigma_J}(\Xi) = -\frac{1}{2}\cdot\sigma_{U_\Xi}^{\mathrm{t}} - \frac{1}{2}\cdot\sigma_{V_\Xi}^{\mathrm{t}} + 2\cdot\sigma_{W_\Xi}^{\mathrm{t}} \; .$$

Since Λ and Ξ are integral cycle classes, and Σ_J is integral modulo balanced cycles, it follows that $cv_{\Sigma_J}(\Lambda)$ and $cv_{\Sigma_J}(\Xi)$ are integral cycle classes. Since, moreover, $cv_{\Sigma_J}(\Delta)$ is σ_J^{t}, we see that

$$\sigma_J^{\mathrm{t}} = \left(2\sigma_{W_\Lambda}^{\mathrm{t}} - \frac{1}{2}\cdot\sigma_{U_\Lambda}^{\mathrm{t}} - \frac{1}{2}\cdot\sigma_{V_\Lambda}^{\mathrm{t}}\right) + \left(2\sigma_{W_\Xi}^{\mathrm{t}} - \frac{1}{2}\cdot\sigma_{U_\Xi}^{\mathrm{t}} - \frac{1}{2}\cdot\sigma_{V_\Xi}^{\mathrm{t}}\right) \; ,$$

where the cycles in the brackets are integral.

Multiplying the latter equality by the integral cycle class σ_J^{t} from the right, and using (15), we obtain

$$\mathrm{id} = \left(2e_{\sigma,W_\Lambda} - \frac{1}{2}\cdot e_{\sigma,U_\Lambda} - \frac{1}{2}\cdot e_{\sigma,V_\Lambda}\right) + \left(2e_{\sigma,W_\Xi} - \frac{1}{2}\cdot e_{\sigma,U_\Xi} - \frac{1}{2}\cdot e_{\sigma,V_\Xi}\right) \; .$$

Since σ_J^{t} is integral, the sums in the brackets remain to be integral.

Arguing similarly as in Case 1, we see that $U_\Lambda = V_\Lambda$ and $U_\Xi = V_\Xi$, and we get the equality

$$\mathrm{id} = (2e_{\sigma,W_\Lambda} - e_{\sigma,U_\Lambda}) + (2e_{\sigma,W_\Xi} - e_{\sigma,U_\Xi}) \; .$$

Re-arranging, we obtain

$$3\cdot\mathrm{id}_J = (2e_{\sigma,W_\Lambda} + e_{\sigma,U_\Xi}) + (2e_{\sigma,W_\Xi} + e_{\sigma,U_\Lambda}) \; .$$

Now, since I_{W_Λ} is the whole diagonal in I^2, the endomorphism e_{σ,W_Λ} is nonzero. As (i_0, j_0) is a pair in W_Ξ, and i_0 is $\sigma(j_0)$, we also obtain that e_{σ,W_Ξ} is nonzero. Then, just as in Case 1, applying Lemma 19 we see that the latter equality, in which the sums in each bracket from the right hand side is integral, leads to a contradiction.

This finishes the proof of the theorem. □

5. The transcendental motive of the Fermat sextic in $\mathbb{P}^3$

To give an explicit example, we use the Fermat sextic in $\mathbb{P}^2$ and the arguments borrowed from the proof of Proposition 7 in [7]. Let x, y, z be the homogeneous coordinates in $\mathbb{P}^2$, and consider the Fermat sextic curve

$$C_6 \subset \mathbb{P}^2 \; ,$$

given by the equation

$$x^6 + y^6 + z^6 = 0 \; .$$

Let μ_6 be the group of all 6-th roots of unit in $\mathbb{C}$, and let

$$\mu_6^2 = \mu_6 \times \mu_6$$

be the two-fold product of μ_6. Then μ_6^2 acts on C_6 by the rule

$$(\epsilon^i, \epsilon^j)(a : b : c) = (\epsilon^i a : \epsilon^j b : c) ,$$

where ϵ is a primitive 6-th root of 1 in $\mathbb{C}$, i.e.

$$\mu_6 = \langle \epsilon \rangle .$$

Since the equation of C_6 is symmetric in all three coordinates, the symmetric group Σ_3 of permutations of three elements acts on C_6 by permuting the coordinates on C_6. Then both groups μ_6^2 and Σ_3 are subgroups in $\mathrm{Aut}(C_6)$ and, moreover,

$$\mathrm{Aut}(C_6) = \mu_6^2 \rtimes \Sigma_3 ,$$

i.e. the group of all regular automorphisms of the curve C_6 is the semidirect product of these two subgroups μ_6^2 and Σ_3, see the main theorem in [33].

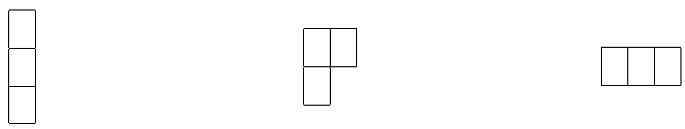
As suggested on page 108 in [7], we look at the global section

$$\omega = \frac{xdy - ydx}{z^5} = \frac{ydz - zdy}{x^5} = \frac{zdx - xdz}{y^5}$$

of the sheaf

$$\Omega_{C_6}(-3) .$$

The three irreducible representations



of Σ_3 and the standard method of constructing irreducible representations of the semidirect product, see Section 9.2 in [28], shows us that the induced action of the automorphism group $\mathrm{Aut}(C_6) = \mu_6^2 \rtimes \Sigma_3$ on $H^0(\Omega_{C_6})$ has three irreducible representations

$$V_{1,1,1} , \quad V_{2,1,0} , \quad V_{3,0,0} ,$$

where $V_{1,1,1}$ is of dimension 1 and generated by the form

$$xyz \cdot \omega ,$$

the space $V_{3,0,0}$ is 3-dimensional and spanned by the forms

$$x^3 \cdot \omega , \quad y^3 \cdot \omega , \quad z^3 \cdot \omega ,$$

and, finally, the space $V_{2,1,0}$ is of dimension 6 and spanned by the following six linearly independent forms

$$x^2 y \cdot \omega , \quad y^2 x \cdot \omega , \quad x^2 z \cdot \omega , \quad z^2 x \cdot \omega , \quad y^2 z \cdot \omega , \quad z^2 y \cdot \omega .$$

Following [7] we consider the elliptic curve with complex multiplication

$$E = \{v^2 w = u^3 - w^3\}$$

in $\mathbb{P}^2$ with coordinates u, v and w. Affinizing C_6 by z and E by w, we also have the affine curves

$$W_6 = C_6 \cap \mathbb{A}^2 = \{x^6 + y^6 = -1\}$$

in $\mathbb{A}^2$ with coordinates x, y, and

$$U_6 = E \cap \mathbb{A}^2 = \{v^2 = u^3 - 1\}$$

in $\mathbb{A}^2$ with coordinates u, v. As in loc.cit., we consider three regular morphisms

$$\phi_i : C_6 \to E \ , \qquad i = 1, 2, 3 \ ,$$

given on the affine parts by the formulas

$$\phi_1 : W_6 \to U_6 \ ,$$
$$\phi_1(x, y) = (-x^2, y^3)$$

and

$$\phi_2 : W_6 \to U_6 \ ,$$
$$\phi_2(x, y) = \left(\frac{y^4}{\sqrt[3]{4}x^2} , \frac{x^6 - 1}{2x^3} \right) \ .$$

If we change the coordinates in $\mathbb{A}^2$ to have $E \cap \mathbb{A}^2$ being defined by the equation

$$u'^3 + v'^3 + 1 = 0 \ ,$$

then we also have a third morphism

$$\phi_3 : W_6 \to U_6 \ ,$$
$$\phi_3(x, y) = (x^2, y^2) \ .$$

The generator

$$\tau \in H^0(\Omega_E)$$

is locally represented by the form $\frac{du}{v}$ in the (u, v)-coordinates, and by the form $\frac{du'}{v'^2}$ in the (u', v')-coordinates, so that we can loosely write

$$\tau = \frac{du}{v} = \frac{du'}{v'^2} \ .$$

Straightforward computations give

$$\phi_1^* \left(\frac{du}{v} \right) = -\frac{2xdx}{y^3} = -2xy^2 \cdot \omega \in V_1 = V_{2,1,0} \ ,$$

$$\phi_2^* \left(\frac{du}{v} \right) = -\sqrt[3]{2^4} y^3 \cdot \omega \in V_2 = V_{3,0,0}$$

and

$$\phi_3^* \left(\frac{du'}{v'^2} \right) = 2xyz \cdot \omega \in V_3 = V_{1,1,1} \ ,$$

see page 108 in [7].

To be in accordance with the notation of Section 4, let

$$G = \mathrm{Aut}(C_6)$$

be the whole group $\mu_6^2 \rtimes \Sigma_3$, let

$$G_1 = \Sigma_3 \ , \quad G_2 = \{(1, 2, 3), (2, 1, 3), (3, 2, 1)\} \subset \Sigma_3$$

and

$$G_3 = \{\mathrm{id}\} \in \Sigma_3$$

be three subsets in Σ_3, where the latter is considered as a subgroup in G. Then the six global sections

$$\sigma^* \phi_1^*(\tau) \ , \quad \sigma \in G_1 \ ,$$

generate the 6-dimensional vector space V_1, the three global sections

$$\sigma^* \phi_2^*(\tau) \ , \quad \sigma \in G_2 \ ,$$

generate the 3-dimensional vector space V_2, and

$$\phi_3^*(\tau)$$

generate the 1-dimensional space V_3. As in Section 4, let

$$f_1 \ , \ldots , f_6$$

be the six regular morphisms $\phi_1 \sigma$ from C_6 onto E, where σ runs the set G_1, arbitrarily indexed, let

$$f_7 \quad f_8 \ , \quad f_9$$

be the three regular morphisms $\phi_2 \sigma$, where σ runs the set G_2, also indexed in an arbitrarily way, and let

$$f_{10}$$

be the last morphism ϕ_3. If

$$n_i = \deg(f_i)$$

then

$$n_i = 6 \qquad \text{for} \quad i = 1, \ldots , 6 \ ,$$
$$n_i = 24 \qquad \text{for} \quad i = 7, 8, 9$$

and

$$n_{10} = 4 \ .$$

Then we have 10×10 projectors Θ_{ij}, and the corresponding transcendental motives T_{ij}, $i, j \in I$, where I be the set $\{1, \ldots , 10\}$. Since $g = 10$, it is easy to compute that

$$\dim(M^2_{\mathrm{tr}}(C_6 \times C_6)) = 200 \ .$$

Applying Theorem 21, we obtain that the transcendental motive $M^2_{\mathrm{tr}}(C_6^2)$ is integrally indecomposable.

Now we are ready to prove Theorem B in Introduction.

Theorem 22. *Let S_6 be the Fermat sextic in $\mathbb{P}^3$ given by the equation*

$$t^6 + u^6 + v^6 + w^6 = 0$$

in $\mathbb{P}^3$. The transcendental motive $M^2_{\mathrm{tr}}(S_6)$ is integrally indecomposable.

Proof. By Lemma 13, without loss of generality one can assume that the ground field k contains the extension $\mathbb{Q}[\sqrt{-1}]$. Recall the following well-known construction from [32]. Let x_1, y_1, z_1 be homogeneous coordinates in $\mathbb{P}^2$, let x_2, y_2, z_2 be homogeneous coordinates in a second copy of $\mathbb{P}^2$, and let ε be a 6-th root of -1. Consider the rational map

$$\varphi : C_6^2 \dashrightarrow S_6$$

given by the quadratic forms

$$[x_1z_2 : y_1z_2 : \varepsilon x_2z_1 : \varepsilon y_2z_1] \; ,$$

see page 98 in loc.cit. This rational map is not defined at 6^2 points (R_i, R_j), where

$$R_i = (1 : -\epsilon^i : 0)$$

is a point on C_6 for each index $i = 0, 1, \ldots, 5$. The composition of the blow up

$$\tilde{C}_6^2 \to C_6^2$$

at the points (R_i, R_j) with the rational map φ is regular. The group μ_6 acts on C_6^2 by the rule

$$\epsilon^i((a,b,c),(a',b',c')) = ((a,b,\epsilon^i c),(a',b',\epsilon^i c')) \; ,$$

and the fixed point locus of this action is exactly the set of 6^2 points (R_i, R_j) described above. This is why the action of μ_6 extends to the action on the blow up $\tilde{C}_6^2$. Moreover, the the quotient surface

$$\tilde{S}_6 = \tilde{C}_6^2/\mu_6$$

is smooth, see page 100 in [32]. Since the composition of the blow up morphism from $\tilde{C}_6^2$ to C_6^2 with the rational map φ is regular and μ_6-equivariant on source, it induces a regular morphism from $\tilde{S}_6$ onto S_6, such that the diagram

$$\begin{array}{ccc} \tilde{C}_6^2 & \xrightarrow{\tilde{\varphi}} & \tilde{S}_6 \\ \downarrow & & \downarrow \\ C_6^2 & \overset{\varphi}{\dashrightarrow} & S_6 \end{array}$$

commutes. Here $\tilde{\varphi}$ is the quotient morphism, and the vertical morphism from the right contracts $6+6$ lines on the surface $\tilde{S}_6$ into points on S_6, so that $\tilde{S}_6$ is the blow up of the Fermat sextic S_6 at 12 points, see Lemma 1.6 in loc.cit.

Let $\tilde{\Delta}_0$, $\tilde{\Delta}$ and Δ be the diagonal classes on the surfaces, respectively, $\tilde{S}_6$, $\tilde{C}_6^2$ and C_6^2. Assume the motive $M(\tilde{S}_6)$ decomposes essentially, and consider two essential mutually orthogonal idempotents $\tilde{\Lambda}_0$ and $\tilde{\Xi}_0$, such that

$$\tilde{\Delta}_0 = \tilde{\Lambda}_0 + \tilde{\Xi}_0$$

in $CH^2(\tilde{S}_6 \times \tilde{S}_6)$. The morphism

$$\frac{1}{6} \cdot \Gamma_{\tilde{\varphi}} : M(\tilde{C}_6^2) \to M(\tilde{S}_6)$$

has a section

$$\Gamma_{\tilde{\varphi}}^{\mathrm{t}} : M(\tilde{S}_6) \to M(\tilde{C}_6^2) \; .$$

The correspondences

$$\tilde{\Pi} = \frac{1}{6} \cdot \Gamma_{\tilde{\varphi}}^{\mathrm{t}} \circ \tilde{\Delta}_0 \circ \Gamma_{\tilde{\varphi}} \; ,$$

$$\tilde{\Lambda} = \frac{1}{6} \cdot \Gamma_{\tilde{\varphi}}^{\mathrm{t}} \circ \tilde{\Lambda}_0 \circ \Gamma_{\tilde{\varphi}} \; ,$$

$$\tilde{\Xi} = \frac{1}{6} \cdot \Gamma^{\mathrm{t}}_{\tilde{\varphi}} \circ \tilde{\Xi}_0 \circ \Gamma_{\tilde{\varphi}} \ ,$$

induce the decomposition

$$\tilde{\Pi} = \tilde{\Lambda} + \tilde{\Xi} \ , \tag{22}$$

and the corresponding splitting

$$M_{\tilde{\Pi}} = M_{\tilde{\Lambda}} + M_{\tilde{\Xi}} \ ,$$

where $M_{\tilde{\Pi}}$ can be viewed as the image of the motive $M(\tilde{S}_6)$ under the embedding of $M(\tilde{S}_6)$ into $M(\tilde{C}_6^2)$.

Since $\tilde{C}_6^2$ is the blow up of C_6^2 at a finite collection of points, the motive $M(\tilde{C}_6^2)$ is a direct sum of the motive $M(C_6^2)$ and a finite number of copies of the Lefschetz motive $\mathbb{L}$, and the transcendental motive $M^2_{\mathrm{tr}}(\tilde{C}_6^2)$ can be identified with the transcendental motive $M^2_{\mathrm{tr}}(C_6^2)$. The correspondence $\tilde{\Pi}$ induces a correspondence Π on $C_6^2 \times C_6^2$, and the decomposition (22) in $CH^2(\tilde{C}_6^2 \times \tilde{C}_6^2)$ induces the corresponding decomposition

$$\Pi = \Lambda + \Xi \ ,$$

of Π into two mutually orthogonal projectors in $CH^2(C_6^2 \times C_6^2)_{\mathbb{Q}}$. Moreover, there exist integral correspondences

$$\Pi_0 \ , \ \Lambda_0 \ , \ \Xi_0 \ \in \ CH^2(C_6^2 \times C_6^2) \ ,$$

such that

$$\Pi = \frac{1}{6} \cdot \Pi_0 \ , \quad \Lambda = \frac{1}{6} \cdot \Lambda_0 \quad \text{and} \quad \Xi = \frac{1}{6} \cdot \Xi_0 \ .$$

Let

$$M_{\Pi} = M_{\Lambda} \oplus M_{\Xi}$$

be the corresponding splitting in $\mathsf{C}(k)_{\mathbb{Q}}$.

The surface S_6 is ρ-maximal, see Proposition 7 in [7], whence

$$\dim(M^2_{\mathrm{tr}}(S_6)) = 20 \ .$$

The action of μ_6 on $\tilde{C}_6^2$ extends the action of μ_6 on C_6^2, and $\tilde{S}_6$ is the quotient of $\tilde{C}_6^2$ by μ_6. The standard properties of group action on algebraic cycles (see, for example, Proposition 2.4 in [34]) give us that the motive $M(\tilde{S}_6)$ is μ_6-invariant inside $M(\tilde{C}_6)$. The numerical and homological equivalence for codimension 2 algebraic cycles with coefficients in $\mathbb{Q}$ coincide, see [25]. The group $H^1(C_6)$ splits into $g = 10$ direct summands corresponding to the morphisms f_i, $i = 1, \ldots, 10$. Using the Künneth formula for the appropriate Weil cohomology theory H^*, one can easily show that the action of μ_6 on the numerical motives $\bar{T}_{ij}$ preserve the diagonal sum $\oplus_{i=1}^{g} \bar{T}_{ii}$. Since the dimension of the latter is 20, and the motive $\bar{M}_{\Pi}$ is μ_6-invariant inside $\bar{M}^2_{\mathrm{tr}}(C_6^2)$, we obtain that

$$\bar{M}_{\Pi_{\mathrm{tr}}} = \bigoplus_{i=1}^{10} \bar{T}_{ii}$$

inside $\bar{M}^2_{\mathrm{tr}}(C_6^2)$.

In other words, the transcendental motive of the surface S_6 lives at the diagonal of the transcendental motive of the product $C_6 \times C_6$, if we divide the relevant projectors by 6. Moreover, since the motive $M(C_6^2)$ is an integral direct summand of the motive $M(\tilde{C}_6^2)$, it follows that there are two integral correspondences

$$\Phi_0, \Psi_0 \in CH^2(C_6^2 \times C_6^2)$$

such that the correspondences

$$\Phi = \frac{1}{6} \cdot \Phi_0 \ , \quad \Psi = \frac{1}{6} \cdot \Psi_0$$

are mutually orthogonal idempotents in $CH^2(C_6^2 \times C_6^2)_{\mathbb{Q}}$,

$$\Delta = \Phi + \Psi \ ,$$

and the splitting

$$\bar{M}^2(C_6^2) = \bar{M}_{\Phi_{\mathrm{tr}}} \oplus \bar{M}_{\Psi_{\mathrm{tr}}}$$

cuts out the diagonal $\oplus_{i=1}^{10} \bar{T}_{ii}$ in to two non-zero components.

It means, that we are exactly in Case 1 of the proof of Theorem 21. The only difference is that the mutually orthogonal idempotents Φ and Ψ cutting the diagonal $\oplus_{i=1}^{10} \bar{T}_{ii}$ in to two nontrivial pieces are not integral but rather the divisions of integral correspondences by 6.

Using Lemma 1 and acting in the same way as in Case 1 of the proof of Theorem 21, we obtain four subsets

$$U_\Phi \ , \ U_\Psi \ , \ W_\Phi \ , \ W_\Psi \subset \ I^2 \ ,$$

where

$$I = \{1, \dots, 10\} \ ,$$
$$I^2 = I \times I \ ,$$
$$I^2 = U_\Phi \cup U_\Psi = W_\Phi \cup W_\Psi$$

the unions are disjoint, such that

$$3 \cdot \mathrm{id} = (2e_{W_\Phi} + e_{U_\Psi}) + (2e_{W_\Psi} + e_{U_\Phi}) \ , \tag{23}$$

$$2e_{W_\Phi} + e_{U_\Psi} = \frac{1}{6} \cdot a \ , \tag{24}$$

$$2e_{W_\Psi} + e_{U_\Phi} = \frac{1}{6} \cdot b \ , \tag{25}$$

and the endomorphisms a and b are integral endomorphisms of the Jacobian J.

For simplicity of notation, let U and W be the preimages of U_Ψ and W_Φ respectively under the diagonal map from I to I^2, and let $W' = I \smallsetminus W$ and $U' = I \smallsetminus U$. The equalities 23, 24 and 25 can be now re-written as

$$3 \cdot \mathrm{id} = (2e_W + e_U) + (2e_{W'} + e_{U'}) \ , \tag{26}$$

$$2e_W + e_U = \frac{1}{6} \cdot a \ , \tag{27}$$

$$2e_{W'} + e_{U'} = \frac{1}{6} \cdot b \ , \tag{28}$$

Let also

$$U_{=24} = \{i \in U \mid n_i = 24\}$$

and, similarly,

$$U'_{=24} = \{i \in U' \mid n_i = 24\} \ .$$

Now, since I^2 is the disjoint union of U and U', at least one of the sets $U_{=24}$ or $U'_{=24}$ is nonempty. Suppose first that they both are nonempty. Then, since all together there are 3 idempotents e_i with $n_i = 24$, it follows that either $U_{=24}$ or $U'_{=24}$ consists of one element, say

$$U_{=24} = \{7\} \ .$$

In that case (27) yields

$$2e_W + \frac{1}{24} \cdot e_7^0 = \frac{1}{6} \cdot a \ .$$

Multiplying both sides by 24, we obtain the eqiality

$$e_7^0 = 4 \cdot a - 2 \cdot 24 \cdot e_W \ .$$

Since n_i divides 24 for any index $i \in I$, we obtain that the norm-endomorphism e_7^0 is divisible by 2 in $\mathrm{End}(J)$, which contradicts the criterion 5.3.4 in [8].

Therefore, one out of the two sets $U_{=24}$ or $U'_{=24}$ consists of three numbers 7, 8 and 9, and the second one is empty, say

$$U_{=24} = \{7, 8, 9\} \ \text{ and } \ U'_{=24} = \emptyset \ .$$

Let

$$A = \{i \in U \mid n_i = 4 \text{ or } 6\}$$

and

$$B = U_{=24} = \{i \in U \mid n_i = 24\} \ .$$

Then

$$2e_W + e_A + e_B = \frac{1}{6} \cdot a \ ,$$

which implies

$$24 \cdot 2 \cdot e_W + 24 \cdot e_A + 24 \cdot e_B = 4 \cdot a \ .$$

As n_i divides 24 for any index $i \in I$, and the result of division of 24 by 4 or 6 is even, the latter equality yields

(29) $$e_7^0 + e_8^0 + e_9^0 = 2c$$

for some c from $\mathrm{End}(J)$.

Next, for any two indices i and j from I, the morphisms

$$f_i : C \to E_i \qquad \text{and} \qquad f_j : C \to E_j$$

induce a morphism

$$f_{ij} : C \to E_i \times E_j \ .$$

The image C_{ij} of the morphism f_{ij} is a smooth projective curve of genus 2 whose Jacobian is isogenous to $E_i \times E_j$. Let e_{ij} be the uniquely defined symmetric idempotent in $\mathrm{End}_{\mathbb{Q}}(J)$ corresponding to the factor $E_i \times E_j$ under the isogeny between J and $E_1 \times \ldots E_{10}$ (see Theorem 5.3.2 in [8]). Then

$$e_{ij} = \frac{1}{n_{ij}} \cdot e_{ij}^0 \ ,$$

where e^0_{ij} is the norm-endomorphism of the abelian subvariety $E_i + E_j$ inside J, and

$$n_{ij} = \deg(f_{ij}) \ .$$

Clearly, n_{ij} divides both n_i and n_j.

Since

$$e_{ij} = e_i + e_j$$

in $\mathrm{End}_{\mathbb{Q}}(J)$, we obtain

$$\frac{1}{n_{ij}} \cdot e^0_{ij} = \frac{1}{n_i} \cdot e^0_i + \frac{1}{n_j} \cdot e^0_j \ .$$

Suppose that n_i equals n_j. In such a case the latter equality implies that

$$m \cdot e^0_{ij} = e^0_i + e^0_j \ ,$$

where m is the quotient of $n = n_i = n_j$ by n_{ij}. Since n_{ij} is strictly smaller than n, it follows that $m > 1$.

This happens when i and i are two indices from the set $\{7, 8, 9\}$. For example,

$$e^0_7 + e^0_8 = m \cdot e^0_{78} \qquad (30)$$

and m divides 24. If m is even, then (29) and (30) imply that e_9 is divisible by 2 in $\mathrm{End}(J)$, which contradicts to 5.3.4 in [8]. Therefore, m is odd. Since m divides 24, we see that m must be 3, and we obtain

$$e^0_7 + e^0_8 = 3 \cdot e^0_{78} \ .$$

Similarly,

$$e^0_7 + e^0_9 = 3 \cdot e^0_{79}$$

and

$$e^0_8 + e^0_9 = 3 \cdot e^0_{89} \ .$$

Solving the system of equations

$$\begin{cases} e^0_7 + e^0_8 = 3 \cdot e^0_{78} \\ e^0_7 + e^0_9 = 3 \cdot e^0_{79} \\ e^0_8 + e^0_9 = 3 \cdot e^0_{89} \end{cases}$$

with regard to e^0_7, e^0_8 and e^0_9, we obtain

$$2 \cdot e^0_8 = 3 \cdot w \ , \qquad (31)$$

where

$$w = e^0_{78} - e^0_{79} + e^0_{89}$$

in $\mathrm{End}(J)$. Dividing (31) by 24 yields

$$2 \cdot e_8 = \frac{1}{8} \cdot w \ .$$

Multiplying by 8, we obtain

$$16 \cdot e_8 = w \in \mathrm{End}(J) \ .$$

This contradicts to Proposition 12.1.1 in [8]. This finishes the proof of the theorem. □

6. Two motivic conjectures and cubic hypersurfaces in $\mathbb{P}^5$

In the previous sections we gave the definition of essential indecomposability of the Chow motive of a smooth projective variety, which can be viewed as integral (in)decomposability of the transcendental motive in case of a smooth projective surface over a field. Then we showed examples of surfaces whose transcendental motive is rationally and hence integrally indecomposable. These are abelian surfaces isogenous to the self-products of elliptic curves with complex multiplication (Proposition 14), algebraic $K3$-surfaces whose motives are known to be finite-dimensional, such as the Fermat or Weil quartic surface S_4 in $\mathbb{P}^3$, all in characteristic 0, see Proposition 15 and Remark 16. We proved Theorem 21 (Theorem A in Introduction) leading to an explicit example of a surface, the self-product of the Fermat curve of degree 6, whose motive is rationally decomposable but integrally not. Although in the latter example we used the fact that the surface has maximal Picard rank, we do not think that this is essential regarding the integral indecomoposability property of $M^2_{\mathrm{tr}}(S)$. Finally, we proved Theorem B which asserts that the transcendental motive of the Fermat sextic in $\mathbb{P}^3$ is also integrally indecomposable. The latter striking example suggests that the following motivic conjecture may be true.

Motivic indecomposability conjecture. *The transcendental motive of a smooth projective surface over a field of characteristic* 0 *is integrally indecomposable.*

This is, of course, a motivic analog of the Hodge-theoretic indecomposability conjecture due to Kulikov, [24], which is known to be false for the Fermat sextic in $\mathbb{P}^3$, see [2].

We will also need another motivic conjecture due to Kimura and O'Sullivan, which asserts that all motives in $\mathsf{C}(k)_{\mathbb{Q}}$ are finite-dimensional, see [1]. Notice that this conjecture is verified only for motives of abelian type, i.e. objects of the full subcategory in $\mathsf{C}(k)_{\mathbb{Q}}$ additively and tensorially generated by motives of curves, see [22]. Our aim is now to show that if the motivic indecomposability conjecture is true, and if the motives of all smooth projective surfaces are finite-dimensional, i.e. the Kimura-O'Sullivan conjecture is true for surfaces, then a very general cubic fourfold in $\mathbb{P}^5$ is not rational.

So let X be a smooth cubic fourfold hypersurface in $\mathbb{P}^5$ over an algebraically closed field k of zero characteristic. Since $\deg(X) < 5$, the hypersurface X is rationally connected, whence

$$CH_0(X)_{\mathbb{Q}} = \mathbb{Q} \; .$$

Fix a point P_0 on X. Then

$$\pi_0 = [P_0 \times X] \; ,$$

$$\pi_1 = 0 \; ,$$

$$\pi_2 = \frac{1}{3} \cdot \gamma^3 \times \gamma \; ,$$

$$\pi_3 = 0 \; ,$$

$$\pi_4 = \Delta_X - \sum_{\substack{i=0 \\ i\neq 4}}^{8} \pi_i \quad \text{(no explicite construction)} ,$$

$$\pi_5 = 0 ,$$

$$\pi_6 = \frac{1}{3} \cdot \gamma \times \gamma^3 ,$$

$$\pi_7 = 0$$

and

$$\pi_8 = [X \times P_0] .$$

This gives the corresponding splitting

$$M(X) = \mathbb{1} \oplus \mathbb{L}^2 \oplus M^4(X) \oplus \mathbb{L}^6 \oplus \mathbb{L}^8$$

in $\mathsf{C}(k)_{\mathbb{Q}}$.

Let ρ_2 be the rank of the algebraic part in $H^4(X)$, for the smooth cubic hypersurface X in $\mathbb{P}^5$. Choosing 2-cycles

$$D_1, \ldots, D_{\rho_2} ,$$

and their Poincaré dual cycles

$$D'_1, \ldots, D'_{\rho_2} ,$$

exactly in the same way as we do it for surfaces, one can easily construct the splitting

$$M^4(X) = M^4_{\mathrm{alg}}(X) \oplus M^4_{\mathrm{tr}}(X) ,$$

in $\mathsf{C}(k)_{\mathbb{Q}}$, where

$$M^4_{\mathrm{alg}}(X) = \mathbb{L}^{\oplus \rho_2} ,$$

i.e.

$$\pi^4_{\mathrm{alg}} = \sum_{i=1}^{\rho_2} [D_i \times D'_i] .$$

Clearly, each copy of the Lefschetz motive $\mathbb{L}$ is the motive $(X, D_i \times D'_i, 0)$, and the transcendental motive $M^4_{\mathrm{tr}}(X)$ is given by the projector

$$\pi^4_{\mathrm{tr}} = \pi_4 - \pi^4_{\mathrm{alg}} .$$

Let also

$$\pi^4_{\mathrm{prim}} = \Delta_X - \frac{1}{3} \cdot \sum_{j=0}^{4} \gamma^{4-j} \times \gamma^j ,$$

and let

$$M^4_{\mathrm{prim}}(X) = (X, \pi^4_{\mathrm{prim}}, 0)$$

be the primitive part of the motive $M(X)$, see [23]. If the cubic $X \subset \mathbb{P}^5$ is very general, the results in [39] show that

$$\rho_2 = 1 ,$$

whence

$$M^4_{\mathrm{prim}}(X) = M^4_{\mathrm{tr}}(X) .$$

Then, for a very general cubic X, we get

$$M^4(X) = \mathbb{L}^{\oplus \rho_2} \oplus M^4_{\mathrm{prim}}(X) .$$

Moreover, if X is very general, then

$$\mathrm{End}_{\mathbb{Q}}(H^4(X)_{\mathrm{prim}}) = \mathbb{Q} ,$$

i.e. the rational Hodge structure on the middle primitive cohomology is indecomposable, see Remark 2.6(a) in [38] and Lemma 5.1 in [37].

Notice that if we could know that the motive $M(X)$ is finite-dimensional, the absence of phantom submotives in finite-dimensional motives would guarantee that the motive $M^4_{\mathrm{tr}}(X)$ is rationally, a fortiori, integrally indecomposable.

Theorem 23. *If the motivic indecomposability conjecture is true, and if the motive of any smooth projective surface is finite-dimensional, then a very general cubic fourfold hypersurface in $\mathbb{P}^5$ is not rational.*

Proof. So, let again X be a very general cubic hypersurface in $\mathbb{P}^5$ over $\mathbb{C}$. Suppose that X is rational, and consider the corresponding birational map

$$\mathbb{P}^4 \dashrightarrow X .$$

Resolving the indeterminacy locus, we get a regular dominant morphism

$$f : Y \to X$$

over k, where Y is obtained by a chain of blow up operations at points, curves and surfaces, starting from $\mathbb{P}^4$.

A crucial geometric argument is this. Let

$$F = F(X)$$

be the Fano variety of the cubic X. By the result of Voisin, there exists a surface

$$F_0 \subset F ,$$

such that any two points on F_0 are rationally equivalent on the fourfold F, see [35]. Moreover, for any line L on X, such that its class $[L]$ in F sits on the surface F_0, the triple line $3L$ is rationally equivalent to the third intersection power,

$$[3L] = \gamma^3 ,$$

of the general hyperplane section γ of the cubic X, see Lemma A.3(v) in [29]. It follows that the class γ of the hyperplane section in $CH^1(X)$ is divisible by 3. Therefore, the splitting

$$M(X) = \mathbb{1} \oplus \mathbb{L}^2 \oplus M^4(X) \oplus \mathbb{L}^6 \oplus \mathbb{L}^8$$

is integral.

The morphism f is generically $1:1$ and dominant. Therefore, the composition $\Gamma_f \circ \Gamma_f^{\mathrm{t}}$ is the identity automorphism of $M(X)$ in the integral category $\mathsf{C}(k)$. In other words, f yields the embedding

$$f^* = \Gamma_f^{\mathrm{t}} : M(X) \to M(Y) ,$$

which integrally splits $M(X)$ from $M(Y)$, and therefore

$$M(Y) = f^*(M(X)) \oplus N$$

in $\mathsf{C}(k)$, where $f^*(M(X))$ is the submotive in $M(Y)$ cut out by the projector $\Gamma_f^{\mathrm{t}} \circ \Gamma_f$ on Y.

Suppose we sequentially blow up s_0 points, s_1 curves $C_1, \ldots, C_{s_1}$ and s_2 surfaces $S_1, \ldots, S_{s_2}$ over k. Then the latter motive splits integrally as

$$M(Y) = M(\mathbb{P}^4) \oplus M_0 \oplus M_1 \oplus M_2 \ ,$$

where

$$M_0 = \oplus_{i=1}^{s_0} (\mathbb{L} \oplus \mathbb{L}^2 \oplus \mathbb{L}^3) \ ,$$

$$M_1 = (\oplus_{i=1}^{s_1} M(C_i)) \otimes (\mathbb{L} \oplus \mathbb{L}^2)$$

and

$$M_2 = \oplus_{i=1}^{s_2} M(S_i) \otimes \mathbb{L} \ .$$

As it was shown in [24], there exists an index $i_0 \in \{1, \ldots, s_2\}$, such that the pullback under the morphism f of the transcendental Hodge structure of the cubic X, being twisted by 1, is an integral sub-Hodge structure in the transcendental Hodge structure of S_{i_0}. More importantly, this integral sub-Hodge structure does not equal to the whole transcendental Hodge structure of S_{i_0}.

Next, the integral splitting

$$\bar{M}(Y) = f^*(\bar{M}(X)) \oplus \bar{N} \tag{32}$$

induces the integral splitting

$$\bar{M}^2_{\mathrm{tr}}(S_{i_0}) = (f^*(\bar{M}^4_{\mathrm{prim}}(X)) \otimes \mathbb{T}) \oplus (\bar{N}_{i_0} \otimes \mathbb{T}) \tag{33}$$

in the category $\mathsf{N}(k)_{\mathbb{Q}}$.

The motives of curves are finite-dimensional by Theorem 4.2 in [22]. Since we assume that the motives of smooth projective surfaces are finite-dimensional, we have in particular that the motives $M(S_i)$ are all finite-dimensional. Then the motive $M(Y)$ is finite-dimensional, and, of course, the motive $M(X)$ is also finite-dimensional.

As the cubic X is very general in $\mathbb{P}^5$,

$$\bar{M}^4_{\mathrm{tr}}(X) = \bar{M}^4_{\mathrm{prim}}(X) \ ,$$

and this motive is indecomposable by Lemma 5.1 in [37] and the absence of phantom submotives in finite-dimensional motives, which is due to Kimura's Proposition 7.5 in [22]. Lemma 3 in [24] gives that

$$\bar{N}_{i_0} \neq 0 \ ,$$

so that both summands in (33) are nontrivial.

In terms of correspondences, the splitting (32) induces an essential decomposition

$$\bar{\Delta} = \bar{\Lambda} + \bar{\Xi}$$

of the diagonal class $\bar{\Delta}$ into two orthogonal idempotents in $N^2(S_{i_0} \times S_{i_0})$, such that

$$\bar{\pi}^2_{\mathrm{tr}}(S_{i_0}) = \bar{\Lambda}_{\mathrm{tr}} + \bar{\Xi}_{\mathrm{tr}}$$

in $\mathrm{End}_{\mathbb{Q}}(\bar{M}^2_{\mathrm{tr}}(S_{i_0}))$,

$$f^*(\bar{M}^4_{\mathrm{prim}}(X)) \otimes \mathbb{T} = M_{\bar{\Lambda}} \qquad \text{and} \qquad \bar{N}_{i_0} \otimes \mathbb{T} = M_{\bar{\Xi}}$$

in $\mathsf{N}(k)_{\mathbb{Q}}$.

Since the motive $M(S_{i_0})$ is finite-dimensional, all numerically trivial endomorphisms of $M(S_{i_0})$ are nilpotent by Proposition 7.5 in [22]. The standard lifting idempotent property gives that there exist two orthogonal idempotents

$$\Lambda' \; , \; \Xi' \in CH^2(S_{i_0} \times S_{i_0}) \; ,$$

such that

$$\bar{\Lambda}' = \bar{\Lambda} \; , \quad \bar{\Xi}' = \bar{\Xi}$$

and

$$\Delta = \Lambda + \Xi$$

in $CH^2(S_{i_0} \times S_{i_0})$. Therefore, we may assume that Λ and Ξ are orthogonal idempotents from the very beginning. In such a case,

$$\pi^2_{\mathrm{tr}}(S_{i_0}) = \Lambda_{\mathrm{tr}} + \Xi_{\mathrm{tr}}$$

in $\mathrm{End}_{\mathbb{Q}}(M^2_{\mathrm{tr}}(S_{i_0}))$, and we obtain the corresponding integral decomposition

$$M^2_{\mathrm{tr}}(S_{i_0}) = M_{\Lambda} \oplus M_{\Xi}$$

in $\mathsf{C}(k)_{\mathbb{Q}}$, such that

$$\bar{M}_{\Lambda} = M_{\bar{\Lambda}}$$

and

$$\bar{M}_{\Xi} = M_{\bar{\Xi}} \; .$$

Since these two numerical motives are nontrivial, we get a contradiction with the indecomposability assumption. □

Remark 24. As it was rightly pointed out to me by Alexander Kuznetsov and Mingmin Shen, it is essential that in Theorem 23 we have to assume motivic finite-dimensionality and integral indecomposability of $M^2_{\mathrm{tr}}(S)$ for all smooth projective surfaces S over $\mathbb{C}$, not only for surfaces in $\mathbb{P}^4$. The reason for that is that when we sequentially blow up points, curves and surfaces, starting from $\mathbb{P}^4$, each next centre of blowing up is contained in the result of the preceding blow up. Therefore, even if the next center is a surface S, a priori S can be contained in the exceptional divisor of the preceding blow up at a point or curve, in which case the projection of S to $\mathbb{P}^4$ is not a surface in $\mathbb{P}^4$.

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